The Finite Representation Property for Representable Residuated Semigroups

Danya Rogozin
Institute for Information Transmission Problems, RAS, Moscow, Russia
Serokell OÜ, Tallinn

Modal and Algebraic Logic and Logical Problems in CS

Relation algebras and their representations

Relation algebra

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Relation algebra

A relation algebra is an algebra $\mathcal{R} = \langle R, 0, 1, +, -, ;, ^{\smile}, \mathbf{1}' \rangle$ such that

- $\langle R, 0, 1, +, - \rangle$ is a Boolean algebra,
- $\langle R, ;, \mathbf{1} \rangle$ is a monoid
- $a^{\smile\smile} = a$,
- $(a + b)^{\smile} = a^{\smile} + b^{\smile}$,
- $(a; b)^{\smile} = b^{\smile}; a^{\smile}$,
- $a^{\smile}; (-(a; b)) \leqslant -b$.

$$(a + b); c = a; c + b; c$$

RA is the class of all relation algebras.

Representable Relation Algebras

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A proper relation algebra (**PRA**) is an algebra $\mathcal{R} = \langle R, 0, 1, \cup, -, ;, ^\smile, \mathbf{1} \rangle$ such that

1. $R \subseteq \mathcal{P}(W)$, where X is a base set, and $W \subseteq X \times X$ is an equivalence relation,
2. $0 = \varnothing$, $1 = W$, $+$ and $-$ are set-theoretic union and complement,
3. ; is the composition of relations, $^\smile$ is the relation converse, $\mathbf{1}$ is the identity subrelation of W.

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A relation algebra $\mathcal{R} \in$ **RA** is *representable* if it is isomorphic to some algebra $\mathcal{R}' \in$ **PRA**. The class of representable relation algebras **RRA** is the closure of **PRA** under isomorphic copies.

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In constrast to Boolean algebras, relation algebras are quite badly behaved due to the following results.

- There exist relation algebras having no representation [Lyndon 1950],
- **RRA** is a variety, but it has no finite axiomatisation [Monk 1964], the equational theory is undecidable [Tarski 1941],
- Representability is undecidable for finite relation algebras [Hirsch, Hodkinson 2001]
- ...

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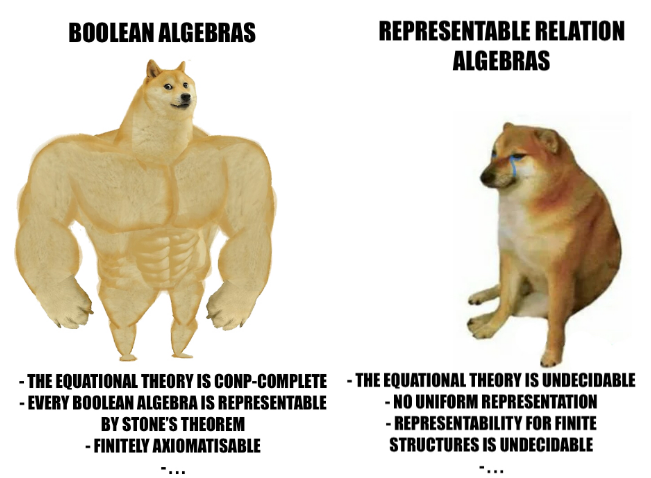

Reducts of relation algebras

Representable reducts of relation algebras

We need a bit of definitions to deal with reducts of representable relation algebras.

Subsignatures of the relation algebra signature

Let τ be a subset of operations and predicates definable in $\mathbf{RA}$. $\mathbf{R}(\tau)$ is the class of subalgebras of τ-subreducts of algebras belonging to $\mathbf{RRA}$. We assume that $\mathbf{R}(\tau)$ is closed under isomorphic copies.

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Examples

- $\mathbf{R}(;, \mathbf{1}, \leqslant)$ — the class of representable ordered monoids,
- $\mathbf{R}(+, \cdot, ;, 0, 1)$ — the class of representable bounded distributive lattice ordered semigroups,
- $\mathbf{R}(;, \smile, \mathbf{dom}, \mathbf{ran}, 0, \leqslant)$ — the class of representable ordered domain algebras.

Finite axiomatisability

- In constrast to the whole class **RRA**, some classes of reducts are finitely axiomatisable.
- Finite axiomatisability of some class is refuted providing a sequence of algebras the ultraproduct of which belongs to its complementary. One may show that with rainbow algebras and back-and-forth games.

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We provide the current results on finite axiomatisability of reducts with the following (incomplete) table:

Yes	No
$\{;, \backslash, /, \leqslant\}$, $\{;, \leqslant\}$, $\{;, \cdot, \mathbf{1}'\}$, $\{;, \smile, \mathsf{dom}, \mathsf{rng}, \leqslant\}$,	$\{;, \leqslant, \mathbf{1}'\}$, $\{;, \cdot, \smile\}$ $\{;, +\}$, $\{;, +, \cdot, \backslash, /\}$,

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There are plenty of open question on finite axiomatisability of varieties generated by representable reducts, etc.

Finite representability

The finite representation property

- A finite representable relation algebra has the finite representation property if it is isomorphic to some $\mathcal{R} \in \mathbf{RRA}$ over a finite base,
- One may transfer this definition for representable reducts.
- Let τ be a subsignature, the class $\mathbf{R}(\tau)$ has the finite representation property if any finite member of $\mathbf{R}(\tau)$ has the FRP.
- The example of a finite relation algebra with no finite representation is the point algebra, an algebra of relations $=$, $>$, and $<$ on the rational line [Maddux 1991], [Hirsch 1995].

Finite representability

Background on the FRP

- Let $\mathbf{R}(\tau)$ be a class of representable reducts, then finite axiomatisability and finite representability of $\mathbf{R}(\tau)$ implies that membership for finite members of $\mathbf{R}(\tau)$ is decidable. Recall that this doesn't hold for all representable algebras due to [Hirsch, Hodkinson 2001],
- Decidability of determining whether an arbitrary finite relation algebra has the representation over a finite base is an open question [Maddux, Hirsch, Hodkinson 2002],
- There are plenty of signatures having (or having no) the FRP, see [Hirsch 2004].

Finite representability

Here is the similar incomplete yes/no table for the finite representation property:

Yes	No
$\{;\,, \mathrm{dom}, \mathrm{rng}, \smile, 0, \mathbf{1'}, \leqslant\},$	$\tau \supseteq \{;\,, \cdot, \mathbf{1'}\}$
$\tau \subseteq \{+, \cdot, -, 0, 1, \smile\},$	$\tau \supseteq \{\cdot, ;\}$
$\{\cdot, ;\,, \mathrm{dom}, \mathrm{rng}\}$	$\tau \supseteq \{;\,, -\}$

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Finite representability is of interest since if $\mathbf{R}(\tau)$ is finitely axiomatisable and has the finite representation property, then the membership problem of $\mathbf{R}(\tau)$ is decidable for finite structures.

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The problems we are interested in are the following:

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Does $\mathbf{R}(;,\backslash,/,\leqslant)$ have the finite representation property? **Yes**

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Residuated semigroups

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The notion of a residuated semigroup

A residuated semigroup is a structure $\mathcal{A} = \langle A, ;, \leqslant, \backslash, / \rangle$ such that

1. $\langle A, ;, \leqslant \rangle$ is a partially ordered semigroup,
2. $\backslash, /$ are binary operations satisfying the residuation property:

$$b \leqslant a \backslash c \Leftrightarrow a; b \leqslant c \Leftrightarrow a \leqslant c / b$$

Residuated semigroups and logic

The logic of residuated semigroups is the Lambek calculus. One may consider this logic from the following perspectives:

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The logic of residuated semigroups is the Lambek calculus. One may consider this logic from the following perspectives:

- Proof-theoretic characterisation of inference in Lambek grammars, the equivalent version of context-free grammars [Pentus 1993],
- Temporal perspective: product and residuals as binary temporal modalities. $\varphi ; \psi$ as "φ and then ψ", etc,
- The Lambek calculus within information processing [van Benthem 1995],
- Modelling of the Minkowski operations in $\mathbb{R}^n$ with the commutative Lambek calculus [G. Bezhanishvili, van Benthem 2007].

Relation algebras and residuals

Residuals are definable in relation algebras as follows:

Residuals in RA

1. $a \backslash b = -(a^{\smile}; -b)$
2. $a / b = -(-a; b^{\smile})$

Residuals in RRA

1. $a \backslash b = \{\langle x, y \rangle \mid \forall z \, (z, x) \in a \Rightarrow (z, y) \in b\}$
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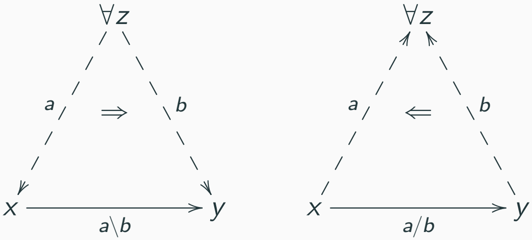
A bit of visualisation:



Representable residuated semigroups

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Residuals in RA

Let A be a set of binary relations on some base set W such that $R = \cup A$ is transitive and W is a domain of R. A relation residuated semigroup is an algebra $\mathcal{A} = \langle A, ;\,, \backslash, /, \leqslant \rangle$ where for each $a, b \in A$

1. $a; b = \{(x, z) \mid \exists y \in W\ ((x, y) \in a\ \&\ (y, z) \in b)\}$,
2. $a \backslash b = \{(x, y) \mid \forall z \in W\ ((z, x) \in a \Rightarrow (z, y) \in b)\}$,
3. $a / b = \{(x, y) \mid \forall z \in W\ ((y, z) \in b \Rightarrow (x, z) \in a)\}$,
4. $a \leqslant b$ iff $a \subseteq b$.

Representable residuated semigroups

- Every residuated semigroup is representable, and, thus, the Lambek calculus is complete w. r. t. $\mathbf{R}(;, \backslash, /, \leqslant)$. This provides relational semantics for the Lambek calculus [Andréka, Mikulás 1994],
- This semantics also has a dynamic interpretation [van Benthem 1995],
- This kind of models is often called R-models.

Problem

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- The problem of finite representability has a surprisingly simple positive solution,
- In particular, that implies that the Lambek calculus is complete w.r.t. finite R-models,
- In fact, we provide an alternative representation for residuated semigroups obeying the finite base requirement.

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Every quantale is a residuated semigroup, residuals are uniquely defined as

1. $a \backslash b = \Sigma\{c \mid a; c \leqslant b\}$,
2. $a / b = \Sigma\{c \mid b; c \leqslant a\}$.

A quantic nucleus

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Quantic nucleus

A *quantic nucleus* on a quantale $\mathcal{Q}$ is a closure operator such that $ja; jb \leqslant j(a; b)$. $a \in \mathcal{Q}$ is j-closed, if $a = ja$. Note that the set of all j-closed elements forms a subquantale where $\Sigma_j A = j(\Sigma A)$ and $a;_j b = j(a; b)$.

The representation theorem for residuated semigroups

Let $\langle X, \leqslant \rangle$ be a poset and $S \subseteq X$, then lS (uS) is the set of all lower (upper) bounds. Let us put $mS = luS$.

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It is cleat that m is a closure operator on a poset $\langle \mathcal{P}(X), \subseteq \rangle$. Let $(\mathcal{P}(X))_m = \{S \subseteq X \mid mS = S\}$, then a poset $\langle (\mathcal{P}(X))_m, \subseteq \rangle$ is complete, where $\Pi S = \bigcap S$ and $\Sigma S = m(\bigcup S)$.

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Then the map $f_m : \langle X, \leqslant \rangle \to \langle (\mathcal{P}(X))_m, \subseteq \rangle$ such that $f_m : x \mapsto \downarrow x$ preserves any existing suprema and infima. Note that this map is well-defined since $\downarrow x$ is m-closed.

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See [Davey, Priestley 2002] to have more details.

The representation theorem for residuated semigroups

Now we extend the construction above for an arbitrary residuated semigroup as follows:

Theorem [Goldblatt, 2006]

Let $\mathcal{S}$ be a residuated semigroup, then $\mathcal{L}$ has an isomorphic embedding into the residuated group of a quantales that preserves existing suprema and infima

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Let us take a look at the proof sketch. Let $\mathcal{S} = \langle S, \leqslant, ; , \backslash, / \rangle$ be a residuated semigroup. The closure operator mX is a quantic nucleus on the free quantale $\langle \mathcal{P}(S), \bullet, \subseteq \rangle$ since $m(X) \bullet m(Y) \subseteq m(X \bullet Y)$.

Thus, $\langle (\mathcal{P}(S))_m, \bullet, \subseteq \rangle$ is a quantale.

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After that one needs to show that an embedding $f_m : \mathcal{S} \to \langle (\mathcal{P}(S))_m, \bullet, \subseteq \rangle$ preserves products and residuals. □

Representation of quantales

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Relational quantale

Let A be a non-empty set. A relational quantale on A is an algebra $\langle R, \subseteq, ; \rangle$, where

1. $R \subseteq \mathcal{P}(A \times A)$,
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3. $;$ is a relational composition that respects all suprema in both coordinates.

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Theorem [Brown, Gurr 1993]

Every quantale $\mathcal{Q} = \langle Q, ;, \Sigma \rangle$ is isomorphic to a relational quantale on Q as a base set.

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$$\hat{a} = \{\langle g, q \rangle \mid g \in \mathcal{G}(\mathcal{Q}), q \in \mathcal{Q}, g \leqslant a; q\} \quad \widehat{\mathcal{Q}} = \{\hat{a} \mid a \in \mathcal{Q}\}$$

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The map $f : a \mapsto \hat{a}$ satisfies the following conditions:

1. $a \leqslant b$ iff $\hat{a} \subseteq \hat{b}$,
2. $\widehat{\Sigma A} = \Sigma \widehat{A}$, $\hat{a}; \hat{b} = \widehat{a; b}$, and $\langle \widehat{\mathcal{Q}}, \subseteq, \Sigma \rangle$ is a complete lattice,
3. $\langle \widehat{\mathcal{Q}}, \subseteq, ; \rangle$ is a relational quantale,
4. $\mathcal{Q}$ is isomorphic to $\langle \widehat{\mathcal{Q}}, \subseteq, ; \rangle$,
5. f is a quantale isomorphism.

The solution for [Problem 19.17, Hirsch, Hodkinson, 2002]

Representation

Let $\tau = \{;, \backslash, /, \leqslant\}$, $\mathcal{A}$ a τ-structure, and X a base set. An interpretation R over a base X maps every $a \in \mathcal{A}$ to a binary relation $a^R \subseteq X \times X$. A representation of $\mathcal{A}$ is an interpretation R satisfying the following conditions:

1. $a \leqslant b$ iff $a^R \subseteq b^R$,
2. $(a; b)^R = \{(x, y) \mid \exists z \in X\ (x, z) \in a^R \ \&\ (z, x) \in b^R\} = a^R; b^R$,
3. $(a \backslash b)^R = \{(x, y) \mid \forall z \in X\ ((z, x) \in a^R \Rightarrow (z, y) \in b^R)\} = a^R \backslash b^R$,
4. $(a \,/\, b)^R = \{(x, y) \mid \forall z \in X\ ((y, z) \in a^R \Rightarrow (x, z) \in b^R)\} = a^R \,/\, b^R$.

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Let $\mathcal{A}$ be a residuated semigroup and $\mathcal{Q}_{\mathcal{A}}$ is a quantale of Galois closed subsets of $\mathcal{A}$. $\widehat{\mathcal{Q}_{\mathcal{A}}}$ is the corresponding relational quantale. Let us define an interpretation $R : \mathcal{A} \to \widehat{\mathcal{Q}_{\mathcal{A}}}$ such that

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Lemma

Let τ be a signature of residuated semigroups. An interpretation $R : \mathcal{A} \to \widehat{\mathcal{Q}_{\mathcal{A}}}$ such that $R : a \mapsto a^R = \widehat{\downarrow a}$ is a τ-representation.

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Corollary

Every residuated semigroup is isomorphic to the subalgebra of some relational quantale.

The solution for [Problem 19.17, Hirsch, Hodkinson, 2002]

Theorem

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Theorem

$\mathbf{R}(;, \backslash, /, \leqslant)$ has the finite representation property.

Let $\mathcal{A}$ be a finite residuated semigroup.

The representation of $\mathcal{A}$ as a subalgebra of a relational quantale clearly belongs to $\mathbf{R}(;, \backslash, /, \leqslant)$. This representation has the form

$$\widehat{\mathcal{A}} = \langle \{\widehat{\downarrow a}\}_{a \in \mathcal{A}}, ;, \backslash, /, \subseteq \rangle.$$

Moreover, such a representation with the corresponding relational quantale has the finite base, if the original algebra is finite. The base set of the quantale $\widehat{\mathcal{Q}_{\mathcal{A}}}$ is the set of Galois stable subsets of $\mathcal{A}$, the cardinality of which is finite.

The $\mathsf{R}(;\,,\cdot)$ and $\mathsf{R}(;\,,+)$ cases

The point algebra

The point algebra might be understood as an algebra of relations on the rational line.

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The point algebra, see, e. g., [Hirsch 1996]

The point algebra $\mathcal{P}$ is a relation algebra consisting of three atoms $\mathbf{1}'$, $<$, $>$ such that $<^{\smile} = >$, $>^{\smile} = <$, $<;< = <$, and $<;> = \mathbf{1}'$.

The point algebra

The point algebra might be understood as an algebra of relations on the rational line.

The point algebra, see, e. g., [Hirsch 1996]

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- The point algebra is frequently used to refute the finite representation property for some representable reducts.

The $\mathsf{R}(;,\cdot)$ case

The failure of the finite representation property for $\mathsf{R}(;,\cdot)$ is shown providing a reduct of the point algebra with no finite representation. Maddux proposed a reduct of the point algebra consting of the elements 0, $<$, and $\mathbf{1}'$.

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The following fact is due to Maddux:

Proposition [Maddux, 2016]

Let U be a non-empty set and $0, \mathbf{1}', < \subseteq U^2$ relations on U. If these relations satisfy the equations below, then $|U| \geqslant \omega$:

1. $<; \mathbf{1}' = < = \mathbf{1}'; <$,
2. $<; < = <$,
3. $<; 0 = 0 = 0; <$,
4. $< \cdot \mathbf{1}' = 0$,

The $\mathsf{R}(;,+)$ case

We are going to prove this fact.

Theorem

$\mathbf{R}(;,+)$ fails to have the finite representation property.

The $\mathsf{R}(;,+)$ case

We use the argument above to refute the finite representation property for $\mathbf{R}(;,+)$. Consider the reduct of the point algebra consising of the following elements.

- $< \, = \{(x, y) \in \mathbb{Q}^2 \mid x < y\}$,
- $\mathbf{1}' = \{(x, y) \in \mathbb{Q}^2 \mid x = y\}$,
- $0 = \varnothing$,
- $\leqslant \, = \{(x, y) \in \mathbb{Q}^2 \mid x \leqslant y\} = \, < + \mathbf{1}'$.

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The operations $+$ and $;$ are defined with the following tables:

$+$	0	$\mathbf{1}'$	$<$	$\leqslant$
0	0	$\mathbf{1}'$	$<$	$\leqslant$
$\mathbf{1}'$	$\mathbf{1}'$	$\mathbf{1}'$	$\leqslant$	$\leqslant$
$<$	$<$	$\leqslant$	$<$	$\leqslant$
$\leqslant$	$\leqslant$	$\leqslant$	$\leqslant$	$\leqslant$

$;$	0	$\mathbf{1}'$	$<$	$\leqslant$
0	0	0	0	0
$\mathbf{1}'$	0	$\mathbf{1}'$	$<$	$\leqslant$
$<$	0	$<$	$<$	$<$
$\leqslant$	0	$\leqslant$	$<$	$<$

The $\mathsf{R}(;,+)$ case

Consider our algebra $\mathcal{A} = \langle\{0, \mathbf{1}', <, \leqslant\}\rangle$. We may introduce the Boolean meet as

$$x \cdot y := \Sigma\{a \in \mathcal{A} \mid a \leqslant x \;\&\; a \leqslant y\}$$

The R$(;,+)$ case

Consider our algebra $\mathcal{A} = \langle\{0, \mathbf{1}', <, \leqslant\}\rangle$. We may introduce the Boolean meet as

$$x \cdot y := \Sigma\{a \in \mathcal{A} \mid a \leqslant x \,\&\, a \leqslant y\}$$

Here is the table for this operation

$\cdot$	0	$\mathbf{1}'$	$<$	$\leqslant$
0	0	0	0	0
$\mathbf{1}'$	0	$\mathbf{1}'$	0	$\mathbf{1}'$
$<$	0	0	$<$	$<$
$\leqslant$	0	$\mathbf{1}'$	$<$	$\leqslant$

And this operation together the composition satisfy Maddux' criterion we discussed above. Thus, the base of any representation of $\mathcal{A}$ is always infinite.

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Thank you for your kind attention!

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