Complexity of finite-variable fragments of products with **K**

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Introduction

Products of propositional modal logics had been introduced in the 1970s:

- Krister Segerberg. Two-dimensional modal logic. Journal of Philosophical Logic, 2(1):77–96, 1973.
- Валентин Борисович Шехтман. Двумерные модальные логики. Математические заметки, 23(5):759–772, 1978.

Introduction

Systematic studies of products appeared in the 1990s:

- Maarten Marx and Yde Venema. Multi-Dimensional Modal Logic, volume 4 of Applied Logic Series. Springer, 1997.
- Dov Gabbay and Valentin Shehtman. Products of modal logics, Part 1. Logic Journal of the IGPL, 6(1):73–146, 1998.

Comprehensive surveys of the fundamental results and techniques:

- Dov Gabbay, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. Many-Dimensional Modal Logics: Theory and Applications, volume 148 of Studies in Logic and the Foundations of Mathematics. Elsevier, 2003.
- Agi Kurucz. Combining modal logics. In P. Blackburn, J. Van Benthem, and F. Wolter, editors, Handbook of Modal Logic, volume 3 of Studies in Logic and Practical Reasoning, pages 869–924. Elsevier, 2008.

Introduction

Products have found applications in

- theoretical computer science;
- temporal-epistemic reasoning;
- knowledge representation;

Also, products are related to predicate modal logics.

Introduction

Complexity of satisfiability and validity problems:

- D. Gabbay and V. Shehtman (1998):
  a number of non-elementary upper bounds;
- M. Marx (1999):
  coNEXPTIME-*hardness for products of two monomodal logics admitting* **S5**-*frames*;
- M. Reynolds, M. Zakharyaschev, R. Hirsch, I. Hodkinson, A. Kurucz, D. Gabelaia, F. Wolter (2001–2005):
  products of two logics of transitive frames, such as **K4**, **S4**, **GL**, *and* **Grz**, *as well as higher-dimensional products, are undecidable*;
- S. Göller, J.-Ch. Jung, and M. Lohrey (2010):
  products of **K** *with* **K**, **K4** *and* $\mathbf{S5}_2$ *are non-elementary* (*but decidable*).

Introduction

Complexity of fragments with limitations on the modal depth:

- M. Marx and S. Mikulás (2001):
  the fragment of $\mathbf{K} \times \mathbf{K}$ *containing formulas of depth at most two is* coNEXPTIME-*complete*;
- S. Göller, J.-Ch. Jung, and M. Lohrey (2010):
  similar results for $\mathbf{K} \times \mathbf{K4}$ *and* $\mathbf{K} \times \mathbf{S5}_2$.

Question

What is the complexity of finite-variable fragments of product logics?

Introduction

It is known:

- a lot of monomodal and polymodal logics are embedable into their one-variable fragments in polynomial time;
- a lot of monomodal and polymodal logics are embedable even into their variable-free fragments in polynomial time;
- a lot of superintuitionistic logics are embedable into their two-variable fragments in polynomial time.

Our aim

Products where at least one factor is **K** *are polynomially embeddable into their single-variable fragments.*

Language

Language contains:

- propositional variables $p_0, p_1, p_2, \ldots$;
- the Boolean constant $\bot$ (falsity);
- the binary Boolean connective $\to$ (implication);
- unary modal connectives $\Box_1, \ldots, \Box_n$.

Formulas are defined as usual: $\varphi \;:=\; \bot \mid p_i \mid (\varphi \to \varphi) \mid \Box_i\varphi$.

We use the standard abbreviations:

$$
\begin{array}{lcllcl}
\neg\varphi & = & (\varphi \to \bot); & \Diamond_i\varphi & = & \neg\Box_i\neg\varphi; \\
\top & = & \neg\bot; & \Box_i^0\varphi & = & \varphi; \\
(\varphi \wedge \psi) & = & \neg(\varphi \to \neg\psi); & \Box_i^{k+1}\varphi & = & \Box_i\Box_i^k\varphi; \\
(\varphi \vee \psi) & = & (\neg\varphi \to \psi); & \Diamond_i^k\varphi & = & \neg\Box_i^k\neg\varphi.
\end{array}
$$

Logics

A normal n-modal logic is a set L of n-modal formulas containing

- the classical tautologies;
- $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$, for every $i \in \{1, \ldots, n\}$,

and closed under

- substitution;
- modus ponens;
- necessitation ($\varphi \in L$ implies $\Box_i \varphi \in L$, for every $i \in \{1, \ldots, n\}$).

We call 1-modal logics monomodal.
The smallest normal monomodal logic is **K**.
Some other standard logics are:

$$
\begin{array}{lcll}
\mathbf{T} & = & \mathbf{K} & \oplus \; \Box p \to p; \\
\mathbf{K4} & = & \mathbf{K} & \oplus \; \Box p \to \Box\Box p; \\
\mathbf{S4} & = & \mathbf{K4} & \oplus \; \mathbf{T}; \\
\mathbf{S5} & = & \mathbf{S4} & \oplus \; p \to \Box\Diamond p.
\end{array}
$$

Kripke semantics

A Kripke n-frame is a tuple $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$, where

- W is a non-empty set of points;
- $R_1, \ldots, R_n$ are binary accessibility relations on W.

A valuation on a frame $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$ is a function v assigning to a propositional variable p a subset $v(p)$ of W.

The truth of formula φ at point x of frame $\mathfrak{F}$ under valuation v is defined recursively:

- $\mathfrak{F}, x \models^v p \quad \leftrightharpoons \quad x \in v(p)$;
- $\mathfrak{F}, x \not\models^v \bot$;
- $\mathfrak{F}, x \models^v \varphi_1 \to \varphi_2 \quad \leftrightharpoons \quad \mathfrak{F}, x \models^v \varphi_1$ implies $\mathfrak{F}, x \models^v \varphi_2$;
- $\mathfrak{F}, x \models^v \Box_i \varphi_1 \quad \leftrightharpoons \quad \mathfrak{F}, y \models^v \varphi_1$ whenever $x R_i y$.

Remark. If φ is variable-free, $\mathfrak{F}, x \models^v \varphi$ if, and only if, $\mathfrak{F}, x \models^{v'} \varphi$, for every pair of valuations v and v'; therefore, we omit valuations when talking about variable-free formulas.

Kripke semantics

- $\mathfrak{F}, x \models \varphi \quad \leftrightharpoons \quad \mathfrak{F}, x \models^v \varphi$, for every valuation v on $\mathfrak{F}$;
- $\mathfrak{F} \models \varphi \quad \leftrightharpoons \quad \mathfrak{F}, x \models \varphi$, for every point x in $\mathfrak{F}$;
- $\mathfrak{F} \models X \quad \leftrightharpoons \quad \mathfrak{F} \models \varphi$, for every $\varphi \in X$.

If $\mathfrak{C}$ is a class of n-frames, $\mathbf{L}(\mathfrak{C})$ denotes the set of n-modal formulas valid on every frame in $\mathfrak{C}$; this set is a normal n-modal logic. The set of monomodal formulas valid on every 1-frame coincides with $\mathbf{K}$.

An n-modal logic L is Kripke complete if $L = \mathbf{L}(\mathfrak{C})$, for some non-empty class $\mathfrak{C}$ of n-frames.

If $W \neq \varnothing$, W' is a non-empty subset of W and $R \subseteq W \times W$, then

- R^* is the reflexive transitive closure of R;
- R^+ is the transitive closure of R;
- R^k, where $k \geqslant 1$, is the k-fold composition of R;
- $R \upharpoonright W' = R \cap (W' \times W')$;
- $R(W') = \{y \in W : xRy, \text{ for some } x \in W'\}$; $R(x)$ means $R(\{x\})$.

Kripke semantics

Let $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$ and $\mathfrak{F}' = \langle W', R'_1, \ldots, R'_n \rangle$ be n-frames. Then,

- $\mathfrak{F}'$ is a subframe of $\mathfrak{F}$, symbolically $\mathfrak{F} \subseteq \mathfrak{F}'$, if $W' \subseteq W$ and $R'_i = R_i \upharpoonright W'$, for each $i \in \{1, \ldots, n\}$;
- $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$, if $\mathfrak{F}' \subseteq \mathfrak{F}$ and $R_i(W') \subseteq W'$, for each $i \in \{1, \ldots, n\}$;
- $\mathfrak{F}'$ is a subframe of $\mathfrak{F}$ generated by $w \in W$ if $W' = (R_1 \cup \ldots \cup R_n)^*(w)$ and $R'_i = W' \upharpoonright R_i$, for each $i \in \{1, \ldots, n\}$, which implies $R_i(W') \subseteq W'$.

It is well known:

- *if $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$ and x is a point in $\mathfrak{F}'$, then, for every φ,*

$$\mathfrak{F}', x \models \varphi \iff \mathfrak{F}, x \models \varphi.$$

Products

The product of 1-frames $\mathfrak{F}_1 = \langle W_1, R_1 \rangle, \ldots, \mathfrak{F}_n = \langle W_n, R_n \rangle$ is an n-frame

$$\mathfrak{F}_1 \times \ldots \times \mathfrak{F}_n \quad = \quad \langle W_1 \times \ldots \times W_n, \bar{R}_1, \ldots, \bar{R}_n \rangle,$$

where, for every $i \in \{1, \ldots, n\}$,

$$\langle x_1, \ldots, x_n \rangle \bar{R}_i \langle y_1, \ldots, y_n \rangle \quad \leftrightharpoons \quad x_i R_i y_i \text{ and } x_k = y_k, \text{ for every } k \neq i.$$

The product of Kripke complete monomodal logics $L_1, \ldots, L_n$ is the n-modal logic

$$L_1 \times \ldots \times L_n \quad = \quad \mathbf{L}(\{\mathfrak{F}_1 \times \ldots \times \mathfrak{F}_n : \mathfrak{F}_1 \models L_1, \ldots, \mathfrak{F}_n \models L_n\}).$$

Modulo renaming the modal connectives, the products of $L_1, \ldots, L_n$, taken in any order, are the same logic. Therefore, when considering products where at least one component is $\mathbf{K}$, we may assume that $L_1 = \mathbf{K}$.

Construction

Let

- φ be an n-modal formula;
- $p_1, \ldots, p_m$ be all the variables of φ;
- p be a propositional variable distinct from $p_1, \ldots, p_m$.

Define

$$\begin{array}{rcl} \alpha & = & \Diamond_1^m \Box_1 \bot; \\ \beta & = & \Diamond_1 \alpha; \end{array}$$

and, for every $k \in \{1, \ldots, m\}$,

$$\gamma_k \quad = \quad \Diamond_1 (\alpha \wedge \Diamond_1^k p).$$

Recursively define the translation σ:

$$\begin{array}{rcll} \sigma(p_k) & = & \gamma_k, & \text{where } k \in \{1, \ldots, m\}; \\ \sigma(\bot) & = & \bot; & \\ \sigma(\psi \to \chi) & = & \sigma(\psi) \to \sigma(\chi); & \\ \sigma(\Box_i \psi) & = & \Box_i (\beta \to \sigma(\psi)), & \text{where } i \in \{1, \ldots, n\}. \end{array}$$

Main lemma

Lemma

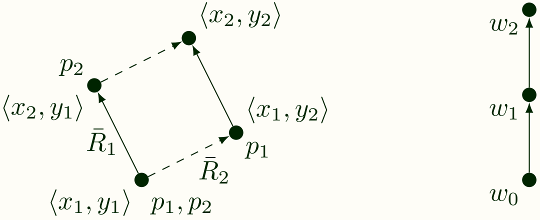
Let $L = \mathbf{K} \times L_2 \times \ldots \times L_n$, for some $L_2, \ldots, L_n$. Then,

$$\varphi \in L \iff \beta \to \sigma(\varphi) \in L.$$

Proof for $(\Leftarrow)$: idea

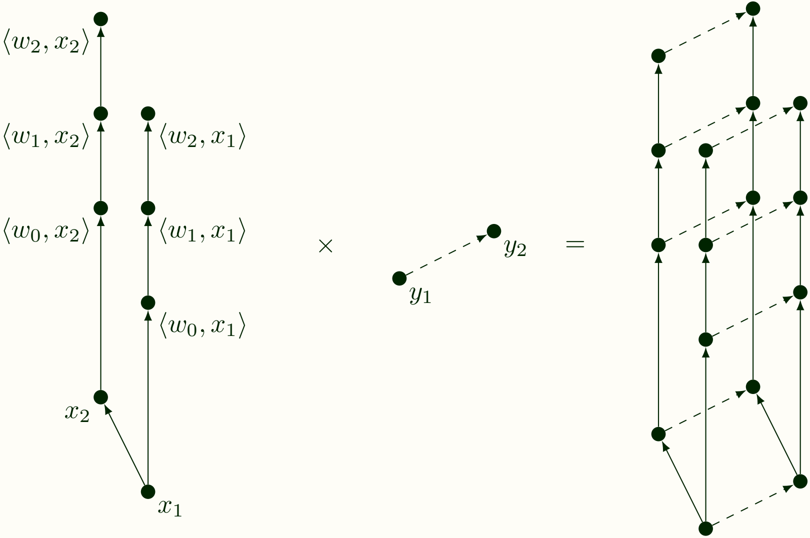
Suppose $\varphi \notin L$; we show that $\beta \to \sigma(\varphi) \notin L$.

Example: $\varphi = p_1 \wedge p_2 \to \Box_1 \neg(\neg p_1 \wedge p_2) \vee \Box_2 \neg(p_1 \wedge \neg p_2)$.

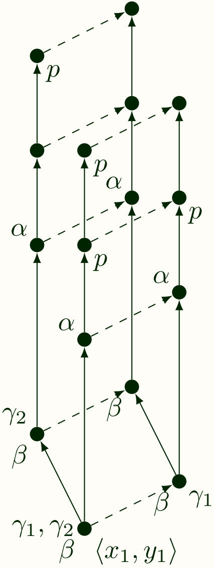


$$\langle x_1, y_1 \rangle \not\models p_1 \wedge p_2 \to \Box_1 \neg(\neg p_1 \wedge p_2) \vee \Box_2 \neg(p_1 \wedge \neg p_2)$$

Proof for ($\Leftarrow$): idea



Proof for ($\Leftarrow$): idea



$$\varphi = p_1 \wedge p_2 \to \Box_1 \neg(\neg p_1 \wedge p_2) \vee \Box_2 \neg(p_1 \wedge \neg p_2)$$

$$\alpha = \Diamond_1^2 \Box \bot$$

$$\beta = \Diamond_1 \alpha$$

$$\gamma_1 = \Diamond_1(\alpha \wedge \Diamond_1^1 p)$$

$$\gamma_2 = \Diamond_1(\alpha \wedge \Diamond_1^2 p)$$

$$\sigma(\varphi) = \gamma_1 \wedge \gamma_2 \to \Box_1(\beta \to \neg(\neg\gamma_1 \wedge \gamma_2)) \vee \Box_2(\beta \to \neg(\gamma_1 \wedge \neg\gamma_2))$$

$$\langle x_1, y_1 \rangle \not\models \beta \to \sigma(\varphi)$$

Proof for ($\Leftarrow$): general construction

Assume $\varphi \notin L$.

Thus, $\mathfrak{F}, \bar{u} \not\models^v \varphi$, for some product $\mathfrak{F} = \langle \bar{W}, \bar{R}_1, \ldots, \bar{R}_n \rangle$ of frames $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ such that $\mathfrak{F}_i = \langle W_i, R_i \rangle \models L_i$, for each $i \in \{2, \ldots, n\}$, some point $\bar{u} \in \bar{W}$ and some valuation v.

We define a product frame $\mathfrak{F}'$ and a valuation v' on $\mathfrak{F}'$ refuting $\beta \to \sigma(\varphi)$.

Let $\mathfrak{F}_\varphi$ be a 1-frame $\langle W_\varphi, R_\varphi \rangle$, where

- $W_\varphi \;=\; \{w_0, w_1, \ldots, w_m\}$;
- $R_\varphi \;=\; \{\langle w_i, w_{i+1} \rangle : 0 \leqslant i < m\}$.

For every $x \in W_1$, let $\mathfrak{F}_\varphi^x = \langle W_\varphi \times \{x\}, R_\varphi^x \rangle$ be an isomorphic copy of $\mathfrak{F}_\varphi$ under the map $f \colon w \mapsto \langle w, x \rangle$.

Proof for ($\Leftarrow$): general construction

Let

- $W_1' \;=\; W_1 \cup (W_\varphi \times W_1)$;
- $R_1' \;=\; R_1 \cup \bigcup\limits_{x \in W_1} R_\varphi^x \cup \{\langle x, \langle w_0, x\rangle\rangle : x \in W_1\}$;
- $\mathfrak{F}_1' \;=\; \langle W_1', R_1'\rangle$.

Thus in $\mathfrak{F}_1'$, for every $x \in W_1$, the root $\langle w_0, x\rangle$ of $\mathfrak{F}_\varphi^x$ is R_1'-accessible from x.

Let $\mathfrak{F}' = \mathfrak{F}_1' \times \mathfrak{F}_2 \times \ldots \times \mathfrak{F}_n = \langle \bar{W}', \bar{R}_1', \ldots, \bar{R}_n'\rangle$.

Thus, $\mathfrak{F} \subseteq \mathfrak{F}'$ and $\mathfrak{F}' \models L$.

Claim: $\mathfrak{F}', \bar{y} \models \beta \iff \bar{y} \in \bar{W}$, *for every* $\bar{y} \in \bar{W}'$.

Proof for ($\Leftarrow$): general construction

Let v' be a valuation on $\mathfrak{F}'$ defined by

$$\bar{z} \in v'(p) \quad \leftrightharpoons \quad \begin{array}{l}\text{there exist } \bar{y} \in \bar{W} \text{ and } k \in \{1, \ldots, m\} \text{ such that} \\ \bar{z} = \langle\langle w_k, y_1\rangle, y_2, \ldots, y_n\rangle \text{ and } \mathfrak{F}, \bar{y} \models^v p_k.\end{array}$$

Claim: *For every subformula θ of φ and every $\bar{x} \in \bar{W}$,*

$$\mathfrak{F}, \bar{x} \models^v \theta \iff \mathfrak{F}', \bar{x} \models^{v'} \sigma(\theta).$$

The proof is by induction on θ.

Since $\mathfrak{F}, \bar{u} \not\models^v \varphi$ and $\bar{u} \in \bar{W}$, we obtain $\mathfrak{F}', \bar{u} \not\models^{v'} \sigma(\varphi)$.

Since $\bar{u} \in \bar{W}$, we obtain $\mathfrak{F}', \bar{u} \models \beta$.

Therefore, $\mathfrak{F}', \bar{u} \not\models^{v'} \beta \to \sigma(\varphi)$.

Since $\mathfrak{F}' \models L$, we obtain $\beta \to \sigma(\varphi) \notin L$.

Proof for ($\Rightarrow$): general construction

Assume $\beta \to \sigma(\varphi) \notin L$.

Thus, $\mathfrak{F}, \bar{u} \not\models^v \beta \to \sigma(\varphi)$, for some product $\mathfrak{F} = \langle \bar{W}, \bar{R}_1, \ldots, \bar{R}_n \rangle$ of frames $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ such that $\mathfrak{F}_i = \langle W_i, R_i \rangle \models L_i$, for each $i \in \{2, \ldots, n\}$, some point $\bar{u} \in \bar{W}$ and some valuation v.

We define a product frame $\mathfrak{F}'$ and a valuation v' on $\mathfrak{F}'$ refuting φ.

Let $W_1' = \{z \in W_1 : \mathfrak{F}_1, z \models \beta\}$.

Since $\mathfrak{F}, \bar{u} \models \beta$ and β does not contain modal operators other than $\Box_1$, clearly $\mathfrak{F}_1, u_1 \models \beta$.

Thus, $u_1 \in W_1'$, and so $W_1' \neq \varnothing$.

Let $R_1' = R_1 \restriction W_1'$ and $\mathfrak{F}_1' = \langle W_1', R_1' \rangle$.

Let $\mathfrak{F}' = \mathfrak{F}_1' \times \mathfrak{F}_2 \times \ldots \times \mathfrak{F}_n = \langle \bar{W}', \bar{R}_1', \ldots, \bar{R}_n' \rangle$.

Since $\mathfrak{F}_1' \subseteq \mathfrak{F}_1$, clearly $\mathfrak{F}' \subseteq \mathfrak{F}$. Also, $\mathfrak{F}' \models L$.

Claim: $\quad \mathfrak{F}, \bar{y} \models \beta \iff \bar{y} \in \bar{W}'$, *for every* $\bar{y} \in \bar{W}$.

Proof for ($\Rightarrow$): general construction

Let v' be a valuation on $\mathfrak{F}'$ defined by

$$v'(p_k) \quad = \quad \{\bar{x} \in \bar{W}' : \mathfrak{F}, \bar{x} \models^v \gamma_k\}.$$

Claim: *For every subformula θ of φ and every $\bar{x} \in \bar{W}'$,*

$$\mathfrak{F}, \bar{x} \models^v \sigma(\theta) \quad \Longleftrightarrow \quad \mathfrak{F}', \bar{x} \models^{v'} \theta.$$

The proof is by induction on θ.

Since $u_1 \in W_1'$, we obtain $\bar{u} \in \bar{W}'$.

Since $\mathfrak{F}, \bar{u} \not\models^v \sigma(\varphi)$, we obtain $\mathfrak{F}', \bar{u} \not\models^{v'} \varphi$.

Since $\mathfrak{F}' \models L$, we obtain $\varphi \notin L$.

Main result

For an n-modal formula φ, let

$$e(\varphi) = \beta \to \sigma(\varphi).$$

Theorem

Let $L = \mathbf{K} \times L_2 \times \ldots \times L_n$.
Then, there exists a polynomial-time embedding of L into its single-variable fragment.

Corollaries

Using results of S. Göller, J.-Ch. Jung and M. Lohrey:

- The single-variable fragments of $\mathbf{K} \times \mathbf{K}$, $\mathbf{K} \times \mathbf{K4}$, and $\mathbf{K} \times \mathbf{S5}_2$ are non-elementary.

Using results of R. Hirsch, I. Hodkinson and A. Kurucz:

- Let L be a logic such that $\mathbf{K} \times \mathbf{K} \times \mathbf{K} \subseteq L \subseteq \mathbf{K} \times \mathbf{S5} \times \mathbf{S5}$. Then, the single-variable fragment of L is undecidable.

Using results of to M. Marx:

- Let L be a logic such that $\mathbf{K} \times \mathbf{K} \subseteq L \subseteq \mathbf{K} \times \mathbf{S5}$. Then, the single-variable fragment of L is coNEXPTIME-hard.
- The single-variable fragment of $\mathbf{K} \times \mathbf{S5}$ is coNEXPTIME-complete.

Corollaries

Consider the following modalities:

- $\mathfrak{F}, x \models^v C\varphi \quad \leftrightharpoons \quad \mathfrak{F}, y \models^v \varphi$ whenever $x(R_1 \cup \ldots \cup R_s)^+ y$;
- $\mathfrak{F}, x \models^v D\varphi \quad \leftrightharpoons \quad \mathfrak{F}, y \models^v \varphi$ whenever $x(R_1 \cap \ldots \cap R_s) y$;
- $\mathfrak{F}, x \models^v A\varphi \quad \leftrightharpoons \quad \mathfrak{F}, y \models^v \varphi$, for every point y in $\mathfrak{F}$.

Claim: *Let $L = L_1 \times L_2 \times \ldots \times L_n$, where L_1 is one of the logics* **PDL**, **CPDL**, **IPDL**, $\mathbf{K}_s$, $\mathbf{K}_s^C$, $\mathbf{K}_s^D$, $\mathbf{K}_s^{CD}$, $\mathbf{K}_s^A$, *where $s \in \mathbb{N}^+$. Then, there exists a polynomial-time embedding of L into its single-variable fragment.*

Using results of D. Gabbay, A.Kurucz, F. Wolter, and M. Zakharyaschev:

Corollary: *Let L be one of the logics* **PDL**, **CPDL**, **IPDL**, $\mathbf{K}_s^A$, $\mathbf{K}_s^C$, $\mathbf{T}_2^C$, $\mathbf{K4}_2^C$, $\mathbf{S4}_2^C$, $\mathbf{KD45}_2^C$, $\mathbf{S5}_2^C$. *Then, single-variable fragments of $\mathbf{K}^A \times L$ and $\mathbf{K}^C \times L$ are undecidable.*

Semiproducts

An n-frame $\mathfrak{G} = \langle \bar{V}, \bar{S}_1, \ldots, \bar{S}_n \rangle$ is an n-ary expanding relativized product of 1-frames $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$, …, $\mathfrak{F}_n = \langle W_n, R_n \rangle$ if $\mathfrak{G} \subseteq \mathfrak{F}_1 \times \ldots \times \mathfrak{F}_n$ and

$$\langle x_1, x_2, \ldots, x_n \rangle \in \bar{V} \text{ and } y \in R_1(x_1) \text{ imply } \langle y, x_2, \ldots, x_n \rangle \in \bar{V},$$

i.e., $\bar{R}_1(\bar{V}) \subseteq \bar{V}$.

Binary expanding relativized products are also known as semiproducts.

We denote by $(\mathfrak{F}_1 \times \ldots \times \mathfrak{F}_n)^{\mathsf{EX}}$ the class of n-ary expanding relativized products of $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$.

The expanding relativized product of Kripke complete monomodal logics $L_1, \ldots, L_n$ is the n-modal logic

$$(L_1 \times \ldots \times L_n)^{\mathsf{EX}} \;=\; \mathbf{L}(\{\mathfrak{F} \in (\mathfrak{F}_1 \times \ldots \times \mathfrak{F}_n)^{\mathsf{EX}} : \mathfrak{F}_i \models L_i, 1 \leqslant i \leqslant n\}).$$

Expanding relativized products of two monomodal logics are also known as semiproducts.

Semiproducts

Theorem

Let $L = (L_1 \times \ldots \times L_n)^{\mathsf{EX}}$, where $L_s = \mathbf{K}$, for some $s \in \{1, \ldots, n\}$. Then, there exists a polynomial-time embedding of L into its single-variable fragment.

Proof: take $e(\varphi) = \beta \to \sigma(\varphi)$.

Thanks

Thanks

Thank you for your time!