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Epistemic Models as the Observable Fragments of Kripke Models

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Abstract

Epistemic reading of Kripke models relies on a hidden assumption of common knowledge of the model which is too restrictive in epistemic contexts.

We explore possible worlds epistemic models in their full generality without common knowledge of the model assumptions and show that an epistemic model can be identified as an "observable fragment" of some Kripke model. We sketch a corresponding theory and argue that it offers a new level of conceptual clarity in epistemic modeling.

Similar analysis applies to intuitionistic models.

Motivations

Williamson, 2015:

Informally, we interpret W as a set of mutually exclusive, jointly exhaustive worlds or states, ... R is a relation of epistemic accessibility: a world w has R to a world x if and only if ... whatever the agent knows in w is true in x.

$$\Big(S.A. : wRx \iff \forall p\, [\ \mathbf{K}p \in w \Rightarrow p \in x].\Big) \tag{1}$$

We define a function $\mathbf{K}$ *from propositions to propositions by the following equation for all propositions p:*

$$\mathbf{K}p = \{w \in W : \forall x \in W, wRx \Rightarrow x \in p\}. \tag{2}$$

In other words, $\mathbf{K}p$ *is true at a world if and only if p is true at every world epistemically accessible from that one. Informally,* $\mathbf{K}p$ *is interpreted as the proposition that the agent knows p.*

Motivations: mismatch in foundations

We will write conventional $u \models X$ instead the set-theoretical $X \in u$.

Obviously, (1) yields

$$u \models \mathbf{K}F \quad \Rightarrow \quad R(u) \models F \tag{3}$$

However, (1) does not warrant the converse:

$$R(u) \models F \quad \Rightarrow \quad u \models \mathbf{K}F, \tag{4}$$

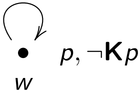
which is built into definition (2). Given knowledge assertions at the states, we can find accessibility relation R by (1), and then determine the knowledge modality $\mathbf{K}$ by (2). However, this $\mathbf{K}p$ is different from the original knowledge assertion "p is known."

The assumpion that knowledge in a possible worlds structure obeys the Kripkean condition (2) is not justified by the adopted meaning of epistemic accessibility (1).

First examples

Example 1.

Consider S5 with a single propositional letter p. Consider also a structure $\mathcal{M}_1$ consisting of one state w generated by $\Gamma = \{p, \neg\mathbf{K}p\}$.[1]



Model $\mathcal{M}_1$.

$\mathbf{K}p$ is false at w. On the other hand, by (2), $\mathbf{K}p$ ought to be true at w. So definitions (1) and (2) do not match in $\mathcal{M}_1$. So, this possible/real world does not have right to exist as a Kripkean epistemic model.

[1] Γ is a complete set of formulas, i.e., for each F, either Γ proves F or Γ proves $\neg F$, and w is the set of formulas derivable from Γ.

Conceptual example

Example 2.

The downside of Kripke specification is that in order to model ignorance of a fact F, one has to commit to a hypothetic world at which F is false. Such a world may not exist.

> *An agent knows the axioms of Peano Arithmetic* PA*, but does not know a theorem F for which a proof has not yet been found. In a Kripke model, we have to have a world v deemed possible by the agent at which $\neg F$ holds. However, there cannot be such a world v because all axioms of* PA *should be true at v and* $\mathrm{PA} \cup \{\neg F\}$ *is inconsistent.*

A possible way out of this predicament is by epistemic models which naturally allow F to be true at each possible world but yet remain unknown.

Observable models, formally

Definition 3.

An *observable model, OM,* over an n-agent logic with knowledge modalities $\mathbf{K}_1, \ldots, \mathbf{K}_n$[2] is a tuple $(W, R_1, \ldots, R_n, \models)$ in which

- W is a nonempty set elements of which are called states (possible worlds);
- $\models$ is a complete truth relation at each world respecting the base logic: for each $u \in W$ the set of formulas true at u is a maximal consistent set over the base logic.
- $u \models \mathbf{K}_i F \;\Rightarrow\; R_i(u) \models F$
  for each formula F, world u and $i = 1, \ldots, n$.

Note that the converse

$$R_i(w) \models F \quad \Rightarrow \quad w \models \mathbf{K}_i F$$

is not postulated!

[2] $S5_n$ is the default, but many other modal logics fit to the same scheme.

Associated Kripke models

A Kripke model $(W, R_1, \ldots, R_n, \Vdash)$ *associated with* $(W, R_1, \ldots, R_n, \models)$ is a Kripke model with the frame $(W, R_1, \ldots, R_n)$ and atomic forcing relation "$\Vdash$":

$$u \Vdash p \quad \textit{iff} \quad u \models p.$$

Definition 4.

A model $(W, R_1, \ldots, R_n, \models)$ is *fully observable* if for each i, w, F,

$$R_i(w) \models F \;\; \Rightarrow \;\; w \models \mathbf{K}_i F.$$

Example 5.

Observable model $\mathcal{M}_1$ from Example 1 is not fully observable.

An observable model $\mathcal{M}$ *coincides with its associated Kripke model iff* $\mathcal{M}$ *is fully observable*

Common knowledge of the model

So, Kripke models are exactly **fully observable models**. Here is an informal[3] sufficient condition under which an observable model is a Kripke model (a fully observable model):

Kripke models are observable models **commonly known to all agents***.*

Once F holds everywhere in $R(u)$, the agent knows this and, knowing R, can conclude that F holds at all states epistemically possible in u, thus coming to justified (by virtue of this argument) knowledge of F. So, in Kripke models, knowledge of F at u, given that F holds in $R(u)$, does not appear from nowhere. A justification for such a knowledge

$$u \Vdash \mathbf{K}F$$

is merely assumed knowledge of the model itself relativized to a specific state u.

[3] It appears that a natural formalization of this condition leads us beyond the current level of propositional modal logic.

Knowing accessibility relations

Definition 6.
An observable model $(W, R_1, \ldots, R_n, \models)$ is *induced* if for all i, u, v, F,

$$uR_iv \quad \Leftrightarrow \quad \textit{for all } F \ (u \models \mathbf{K}_i F \ \Rightarrow \ v \models F). \qquad (5)$$

Informally, an induced OM $(W, R_1, \ldots, R_n, \models)$ has all possible accessibility relations given $(W, \models)$. Since, in canonical models, accessibility relations R_i's satisfy (5), *all canonical models are induced*.

On the structure of induced observable models

Let $(W, R_1, \ldots, R_n, \models)$ be an induced observable model over $S5^n$.

Proposition. Each R_i *is an equivalence relation on* W*.*

The intuition of indistinguishability is similar to Kripke models: we can interpret $R(w)$ as some set of states indistinguishable from w by facts known to the agent. The principal difference between OM's and Kripke models is that in the former, a validity of F in $R(w)$ does not yield knowledge of F: there is room for ignorance of agents about valid facts[4].

Corollary 7.

$R(w) \models \mathbf{K}F$ *or* $R(w) \models \neg\mathbf{K}F$*.*

Indeed, suppose $w \models \mathbf{K}F$ but for some $x \in R(w)$, $x \models \neg\mathbf{K}F$. By negative introspection, $x \models \mathbf{K}\neg\mathbf{K}F$, hence $R(x) \models \neg\mathbf{K}F$. Since, by Proposition, $R(x) = R(w)$, $w \models \neg\mathbf{K}F$. A contradiction.

[4]This as a feature that makes observable models more flexible and realistic.

Derivations from hypotheses in epistemic logic

The Necessitation rule:

$$\vdash F \quad \Rightarrow \ \vdash \mathbf{K}_i F.$$

is not epistemically valid in a general setting for derivations from assumptions: for some Γ, $\Gamma \vdash F$ does not yield $\Gamma \vdash \mathbf{K}_i F$.

Definition 8.

For a given set of formulas Γ (here called "hypotheses" or "assumptions") we consider *derivations from* Γ: assume all $S5_n$-theorems together with Γ and use classical reasoning (rule *Modus Ponens*). The notation

$$\Gamma \vdash A$$

represents 'A is derivable from Γ.'

If A is derived from Γ, we cannot conclude that A is known, since Γ itself can be unknown. However, for some "good" Γ's, Necessitation is valid.

Canonical models for S5 with a single letter

Example of canonical model constructions associated to S5 with a single propositional letter p, S5(p). Such models are defined by their possible worlds W and truth relations $\models$ since the accessibility relations are induced and can be recovered from $(W, \models)$.

The modality-free fragment generated by $\{p\}$ admits two possible worlds: one generated by $\{p\}$ and the other generated by $\{\neg p\}$.

S5(p) admits exactly four possible worlds (maximal consistent sets):

- A, generated by $\{\mathbf{K}p\}(= \{p, \mathbf{K}p\})$;
- B, generated by $\{p, \neg\mathbf{K}p\}$;
- C, generated by $\{\neg p, \neg\mathbf{K}\neg p\}$;
- D, generated by $\{\mathbf{K}\neg p\}(= \{\neg p, \mathbf{K}\neg p\})$.

Canonical models for S5 with a single letter

Consistency of each of A–D is straightforward since each has an easy Kripke model. Now we check that each of A–D is complete, i.e., that each proves F or $\neg F$ for any formula F in the language of S5(p).

Completeness of A. First we note that A is closed under Necessitation: $A \vdash F$ yields $A \vdash \mathbf{K}F$. Standard induction on derivations of F. The key point here is that $A \vdash \mathbf{K}A$. Once we establish Necessitation in A, we proceed to proving that for each F, $A \vdash F$ or $A \vdash \neg F$. Induction on F. Obvious for atomic formulas and Boolean connectives. Let $F = \mathbf{K}X$. If $A \vdash X$, then, by Necessitation, $A \vdash \mathbf{K}X$. If $A \vdash \neg X$, then, by reflexivity, $A \vdash \neg\mathbf{K}X$.

Completeness of B. Here Necessitation is not admissible since $B \vdash p$, but $B \nvdash \mathbf{K}p$. We will use the S5-normal forms.

Canonical models for S5 with a single letter

Lemma 9 (S5 normal forms).

In S5, every formula is provably equivalent to a formula in normal form which is a disjunction of conjunctions of type

$$\alpha \wedge \mathbf{K}\beta \wedge \neg\mathbf{K}\gamma_1 \wedge \ldots \wedge \neg\mathbf{K}\gamma_m \tag{6}$$

where $\alpha, \beta, \gamma_1, \ldots, \gamma_m$ are all purely propositional formulas. For S5(p) we may assume that each of them is one of $\top, \bot, p, \neg p$.

We have to check that for each formula F of type (6), $B \vdash F$ or $B \vdash \neg F$.

If $\alpha = \bot, \neg p$, then $B \vdash \neg F$. If $\alpha = \top, p$, then $B \vdash \alpha$: proceed to β.

If $\beta = \bot, \neg p$, then, by reflexivity, $B \vdash \neg F$. If $\beta = p$, then again, $B \vdash \neg F$. If $\beta = \top$, then $B \vdash \mathbf{K}\beta$ and we proceed to γ_i.

If at least one of γ_i is $\top$, then $B \vdash \neg F$. Otherwise, all conjuncts in F are provable in B. Indeed, for $\gamma_i = \bot$, use $B \vdash \neg\mathbf{K}\bot$. For $\gamma_i = p$ use the fact that $\neg\mathbf{K}p \in B$. For $\gamma_i = \neg p$, use reflexivity $p \to \neg\mathbf{K}\neg p$.

Canonical models for S5 with a single letter

Completeness of C. Similar to B

Completeness of D. Similar to A, since D also enjoys Necessitation.

The collection of A–D **exhausts** all possibilities for states over $\mathsf{S5}(p)$.

Indeed, for the remaining four logical possibilities for p and knowledge assertion about p, $\{p, \mathbf{K}\neg p\}$ and $\{\neg p, \mathbf{K}p\}$ are inconsistent and so are any of its extensions.

The last options: $\{p, \neg\mathbf{K}\neg p\} = \{p\} \subset A$ and $\{\neg p, \neg\mathbf{K}p\} = \{\neg p\} \subset C$, generate no new states.

Canonical models for S5 with a single letter: accessibility

$W = \{A, B, C, D\}$ – all possible states over S5(p).

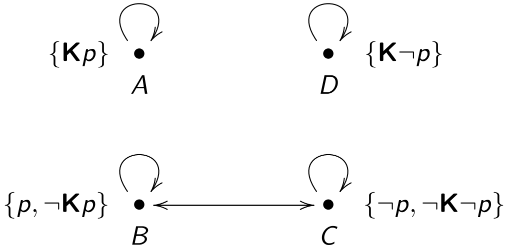
By Proposition, R is an equivalence relation on W. Consider all six possible pairs of different states in W and rule out not accessible ones.

For (A, X) to be in R, p should be in X, which rules out $\{A, C\}$, $\{A, D\}$. Likewise, $\{B, D\}$ are not connected.

Pairs $\{A, B\}$ and $\{C, D\}$ are not connected due to positive introspection, e.g., since $\mathbf{K}p \in A$, $\mathbf{KK}p \in A$ too, hence $(A, X) \in R$ yields $\mathbf{K}p \in X$; this rules out $\{A, B\}$.

The only remaining possibility for R-connection is pair $\{B, C\}$, and they are connected! From the Kripke canonical model perspective, there should be a state in W accessible from B in which $\neg p$ holds, and C is the only remaining possibility, hence $(B, C) \in R$.

Canonical models for S5 with a single letter: picture



Each of fifteen non-empty subsets of $\{A, B, C, D\}$ *is an induced observable model. Seven of them are Kripke models:*

$$\{A\},\ \{D\},\ \{A, D\},\ \{B, C\},\ \{A, B, C\},\ \{B, C, D\},\ \{A, B, C, D\}.$$

Among *OM*'s here is $\{A, B\}$ – all states at which p holds; this is the canonical *OM* model of $\Gamma = \{p\}$ which is therefore not a Kripke model.

From observable models to Kripke models

Each Kripke model is an *OM*, hence *OM*s constitute a well-principled generalization of Kripke models covering more epistemic situations.

How can we build observable models in general? It appears, to define an *OM*, we have to specify complete truth relations at each world, and this is the task which Kripke models do naturally. Can we combine the generality of *OM*s and the convenience of Kripke models?

We show that each observable model is a sub-model of an appropriate Kripke model. This suggests a general method, "scaffolding," of defining *OM*: build a convenient redundant Kripke model, and carve out of it the desired observable model.

Embedding theorem

Theorem 10.
labelembed For any observable model $\mathcal{M} = (W, R_1, \ldots, R_n, \models)$ *there is a Kripke model* $(\widetilde{W}, \widetilde{R_1}, \ldots, \widetilde{R_n}, \Vdash)$ *such that*

a) $W \subseteq \widetilde{W}$*;*
b) $R_i \subseteq \widetilde{R}_i$*;*
c) for each $u \in W$ *and each* F*,* $u \models F$ *iff* $u \Vdash F$*.*

Proof.
Here is a draft proof for the case when all worlds in $\mathcal{M}$ are different as maximal consistent sets[5]. Take

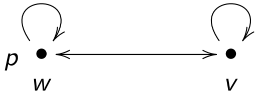
$$(\widetilde{W}, \widetilde{R_1}, \ldots, \widetilde{R_n}, \Vdash)$$

to be the canonical Kripke model $CM(\mathrm{S5}^n)$: each $u \in W$ is also a world in $CM(\mathrm{S5}^n)$ and in both models, $(W, R_1, \ldots, R_n, \models)$ and $CM(\mathrm{S5}^n)$, F holds at u iff $F \in u$. For (b) it suffices to note that uR_iv yields $u\widetilde{R}_iv$. ∎

[5] A general proof is a straightforward adaptation of this argument.

Example 11.

Here is the finite example of such an embedding. The aforementioned observable model $\mathcal{M}_1$ from Example 1 is naturally embedded into a scaffolding model $\mathcal{M}_2$. This will take adding a new world and extending the accessibility relation. $\mathcal{M}_1$ is $\mathcal{M}_2$ restricted to $\{w\}$.

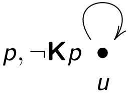


Model $\mathcal{M}_2$.

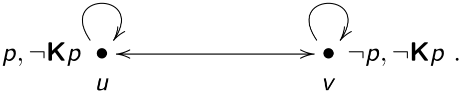
We see here that the Kripke model requires two worlds to emulate a singleton observable model. The blow-up of the number of worlds when embedding an observable model to a Kripke model can be infinite: it is easy to build a singleton OM with only infinite scaffoldings.

An attempt to defend Kripke observable modeling

This observable model: only the “real” world, p is true but unknown



is not a Kripke model, but can be extended to one:



However, according to the epistemic description, world v is not regarded possible by the agent and is an artifact of Kripke modeling rather than an epistemically meaningful possible word. This can make a principal difference for an epistemic analysis.

Cf. the *Ignorant Agent* example (some slides below).

Definition 12.

Let Γ be a set of formulas. By $\Gamma \models F$ we understand the situation when for each observable model $\mathcal{M}$ and its state u,

$$\mathcal{M}, u \models \Gamma \quad \Rightarrow \quad \mathcal{M}, u \models F.$$

Corollary 13.

Soundness and completeness of $\mathsf{S5}^n$*:*

$$\Gamma \vdash F \;\; \textit{iff} \;\; \Gamma \models F.$$

Proof.

Soundness is immediate. Completeness follows from the Kripke completeness. ∎

OMs do not yield new logical consequence relations, or new tautologies. OMs rather offer a fresh view of epistemic modeling by suggesting a new broad class of epistemic models.

Questions: find algebraic, topological, and other versions of Observable Models.

Canonical observable models in a general setting

Definition 14.

A *canonical observable model of a set of formulas* Γ, $CM(\Gamma)$, is the collection of all worlds containing Γ with **induced accessibility relations**.

We show that for many Γ's, the corresponding canonical model $CM(\Gamma)$ is not fully observable, hence not a Kripke model. We give a criterion of when the canonical model of Γ is a Kripke model.

Example 15.

Canonical model $CM(p)$ for $\Gamma = \{p\}$ in S5. There are two possible worlds in $CM(p)$, generated by $\{\mathbf{K}p\}$ (world A) and $\{p, \neg\mathbf{K}p\}$ (world B). Worlds A and B are not connected by the undistinguishability relation R, p holds at both worlds, but is not known in B. The canonical model $CM(p)$ is not a Kripke model, since p holds in $CM(p)$, but $\mathbf{K}p$ does not.

Common knowledge

Consider a representative case of two agents. We will use abbreviations: for "everybody's knowledge"

$$\mathbf{E}X = \mathbf{K}_1 X \wedge \mathbf{K}_2 X,$$

and "common knowledge"

$$\mathbf{C}X = \{X,\ \mathbf{E}X,\ \mathbf{E}^2X,\ \mathbf{E}^3X,\ \ldots\}.$$

As one can see, $\mathbf{C}X$ is an infinite (though decidable) set of formulas. Since modalities $\mathbf{K}_i$ commute with the conjunction $\wedge$, $\mathbf{C}X$ is provably equivalent to the set of all formulas which are X prefixed by iterated knowledge modalities:

$$\mathbf{C}X = \{P_1P_2\ldots P_kX \mid k = 0, 1, 2, \ldots,\quad P_i \in \{\mathbf{K}_1, \mathbf{K}_2\}\}.$$

Naturally,

$$\mathbf{C}\Gamma = \bigcup\{\mathbf{C}F \mid F \in \Gamma\}$$

that represents "Γ is common knowledge."

Common Knowledge and Necessitation

Lemma 16.

A set of formulas Γ is closed under Necessitation if and only if $\Gamma \vdash \mathbf{C}\Gamma$, i.e., Γ proves its own common knowledge.

Proof.

Direction 'if.' Assume $\Gamma \vdash \mathbf{C}\Gamma$ and prove by induction on derivations that $\Gamma \vdash X$ yields $\Gamma \vdash \mathbf{K}_i X$. For X being from $S5_n$, this follows from the rule of Necessitation in $S5_n$. For $X \in \Gamma$, it follows from the assumption that $\Gamma \vdash \mathbf{C}X$, hence $\Gamma \vdash \mathbf{K}_i X$. If X is obtained from *Modus Ponens*, $\Gamma \vdash Y \to X$ and $\Gamma \vdash Y$. By IH, $\Gamma \vdash \mathbf{K}_i(Y \to X)$ and $\Gamma \vdash \mathbf{K}_i Y$. By the distributivity principle of $S5_n$, $\Gamma \vdash \mathbf{K}_i X$.

For 'only if,' suppose that Γ is closed under Necessitation and $F \in \Gamma$, hence $\Gamma \vdash F$. Using appropriate instances of the Necessitation rule in Γ we can derive $P_1 P_2 P_3, \ldots, P_k F$ for each prefix $P_1 P_2 P_3, \ldots, P_k$ with P_i is one of $\mathbf{K}_1, \mathbf{K}_2$. Therefore, $\Gamma \vdash \mathbf{C}F$ and $\Gamma \vdash \mathbf{C}\Gamma$. ∎

Theorem 17.
The following are equivalent:

a) $CM(\Gamma)$ *is fully observable (i.e., a Kripke model);*

b) Γ *admits Necessitation;*

c) Γ *proves its own common knowledge.*

Proof.
(b) is equivalent to (c) by Lemma 16. We now check (a) and (b).

If Γ does not admit Necessitation, there is a formula F such that $\Gamma \vdash F$, but $\Gamma \nvdash \mathbf{K}F$[6]. F holds everywhere in $CM(\Gamma)$ but the set $\Delta = \Gamma \cup \{\neg\mathbf{K}F\}$ is consistent. Consider a maximal consistent extension u of Δ. Obviously, u is a world in $CM(\Gamma)$, $\neg\mathbf{K}F$ holds in u and F holds in $R(u)$ which makes $CM(\Gamma)$ not fully observable and hence not a Kripke model.
(to be contd.) ∎

[6] Here again, we ignore indices i in $\mathbf{K}_i$ and R_i.

Canonical Models of Sets of Assumptions

Proof.

(contd.) Suppose Γ admits Necessitation. To prove that $(CM(\Gamma), \models)$ with the canonical R is a Kripke model, it suffices to establish the so-called Truth Lemma for the associated Kripke model $(CM(\Gamma), R, \Vdash)$: membership in u coincides with the truth value at u

$$F \in u \quad \text{iff} \quad u \Vdash F.$$

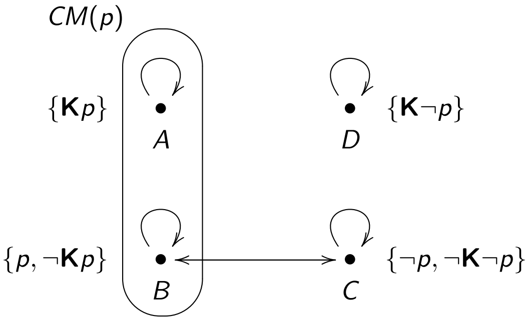
The proof of the Truth Lemma is quite standard, by induction on F. The principal case is $F = \mathbf{K}X \notin u$. By maximality of u, $\neg\mathbf{K}X \in u$. Consider the set $u^{\Box} = \{Y \mid \Box Y \in u\}$. Then the set $u^{\Box} \cup \{\neg X\}$ is consistent; take v to be its maximal consistent extension. To check that $v \in CM(\Gamma)$, it suffices to show that $\Gamma \subseteq v$. Let $\varphi \in \Gamma$. Since Γ admits Necessitation, $\Gamma \vdash \Box\varphi$ hence $\Box\varphi \in u$, $\varphi \in u^{\Box}$, hence $\varphi \in v$. So, uRv, and $\neg X \in v$, hence, by IH, $v \not\Vdash X$ and $u \not\Vdash F$. Theorem 17 is now immediate:

$$u \models F \quad \Leftrightarrow \quad F \in u \quad \Leftrightarrow \quad u \Vdash F.$$

∎

More examples

The canonical model of $\Gamma = \{p\}$, $CM(p)$[7], is the set of worlds at which p holds, i.e., $W = \{A, B\}$. Model $CM(p)$ is not fully observable, not a Kripke model, and is an illustration of Theorem 17, since $\{p\}$ is not closed under Necessitation.



Canonical model $CM(p)$ (in the oval).

[7]We drop brackets in $CM(\{p\})$ and similar cases for better readability.

and more

The canonical model of $\Gamma = \{\mathbf{K}p\}$ is the set of worlds at which $\mathbf{K}p$ holds

$$CM(\mathbf{K}p) = \{A\}.$$

$\{\mathbf{K}p\}$ enjoys Necessitation, its canonical observable model is fully observable, i.e., is a Kripke model.

Let $\Gamma = \{\neg\mathbf{K}p, \neg\mathbf{K}\neg p\}$. By negative and positive introspection, Γ is closed under Necessitation, hence $CM(\Gamma)$ should also be fully observable. Worlds A and D are not compatible with Γ and hence are not in $CM(\Gamma)$. Since none of B and C is such a model (neither is fully observable),

$$CM(\{\neg\mathbf{K}p, \neg\mathbf{K}\neg p\}) = \{B, C\}.$$

Indeed, it is an easy exercise to derive $\neg\mathbf{K}p$ and $\neg\mathbf{K}\neg p$ in B and C. This is a Kripkean situation, i.e., $CM(\{\neg\mathbf{K}p, \neg\mathbf{K}\neg p\})$ is fully observable.

A simple observable model without meaningful scaffoldings

Here is a natural example, *Ignorant Agent*, of a two state *OM* which scaffolding Kripke models are infinite and nonsensical. Consider S4 with a single propositional letter p and a situation in which the agent has no any specific knowledge about the state of nature, i.e., the agent knows only logical truths derived in S4. Consider two sets of formulas:

$$\Gamma^+ = \{p\} \cup \{\neg \mathbf{K}F \mid \text{ S4} \not\vdash F\}, \quad \Gamma^- = \{\neg p\} \cup \{\neg \mathbf{K}F \mid \text{ S4} \not\vdash F\}$$

Lemma 18.

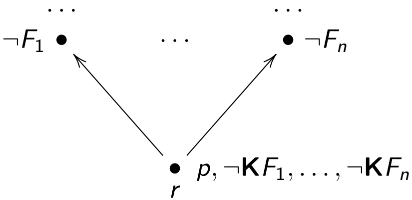
Γ^+ *and* Γ^- *are consistent, complete, and decidable.*

Proof.

Consider Γ^+, the case of Γ^- is similar.

Consistency. Suppose o/w: $\Gamma^+ \vdash \bot$. Then $p, \neg\mathbf{K}F_1, \ldots, \neg\mathbf{K}F_n \vdash \bot$ for some F_i's not provable in S4. By S4, S4 $\vdash p \rightarrow (\mathbf{K}F_1 \vee \ldots \vee \mathbf{K}F_n)$. Since each of F_k's has a countermodel, by a standard S4-model construction, we build an auxiliary S4 Kripke model, at the root node r of which $r \Vdash p$ and $r \nVdash \mathbf{K}F_1 \vee \ldots \vee \mathbf{K}F_n$, a contradiction. (to be contd.) ∎

The auxiliary model

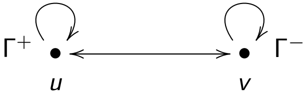


A simple observable model without meaningful scaffoldings

Completeness: For each F, $\Gamma^+ \vdash F$ or $\Gamma^+ \vdash \neg F$. Induction on F. Atomic and Boolean cases are trivial. Let F be $\mathbf{K}X$ for some X. If $\mathsf{S4} \vdash X$, then $\mathsf{S4} \vdash \mathbf{K}X$, $\Gamma^+ \vdash \mathbf{K}X$. If $\mathsf{S4} \not\vdash X$, then $\Gamma^+ \vdash \neg\mathbf{K}X$.

Decidability is immediate by a Post argument.

A natural *OM* over S4 for *Ignorant Agent* is $\mathcal{M}_3$:



It suffices to show that u and v are indeed indistinguishable, i.e., $\Gamma^+ \vdash \mathbf{K}F$ yields $\Gamma^- \vdash F$ (and symmetrically for Γ^- and Γ^+). Suppose $\Gamma^+ \vdash \mathbf{K}F$. Then $\mathsf{S4} \vdash F$, since o/w $\mathsf{S4} \not\vdash F$ and $\neg\mathbf{K}F \in \Gamma^+$ which would make Γ^+ inconsistent. Then $\Gamma^- \vdash F$.

Any scaffolding Kripke model for $\mathcal{M}_3$ is infinite. It should contain counter-worlds for all F's s.t. $\mathsf{S4} \not\vdash F$ which does not make much sense since S4 is decidable and there is no need to carry all these additional worlds to define the situation epistemically.

Intuitionistic models also assume "knowledge of the model"

Nodes in an intuitionistic Kripke model $(W, \preceq, \Vdash)$ are information states in a discovery process and

$$u \preceq v$$

means that v has more truths than u.

In order to use $(W, \preceq, \Vdash)$ as a specification device, **we have to assume the knowledge of the model**. Without this assumption, the fact that F does not hold at all states accessible from w can be unknown and does not yield $\neg F$ at w. Although

$$w \Vdash \neg F \quad \Rightarrow \quad v \nVdash F \textit{ for all } v \textit{ s.t. } w \preceq v,$$

the converse "$\Leftarrow$" is not well-principled.

(New) intuitionistic models as observable fragments

We propose an appropriate adjustment: an **observable fragment** of an intuitionistic Kripke model $\mathcal{K} = (W, \preceq, \Vdash)$ is $\mathcal{O} = (W_O, \preceq_O, \models)$ with

- $W_O \subseteq W$, states from W_O are called "observable";
- $\preceq_O$ is a subset of $\preceq$ restricted to W_O;
- for $v \in W_O$, $v \models F$ is defined as $v \Vdash F$ (the whole model $\mathcal{K}$ is needed to define $\models$ in $\mathcal{O}$).

An intuitionistic observable model is, by definition, an observable fragment of some intuitionistic Kripke model.

(New) intuitionistic models: example

Intuitionistic Kripke model $\mathcal{K}$: $W = \{1, 2\}$, $1 \preceq 2$, $1 \nVdash p$, $2 \Vdash p$.

Intuitionistic observable model $\mathcal{O}$ (circled): $W_O = \{1\}$, $1 \not\models p$.

In $\mathcal{O}$, p does not hold, but yet $1 \nVdash \neg p$.

$\mathcal{O}$ is not a Kripke model!

Can Kripke models do the same job as observable models?

An epistemic situation can be regarded as a (global) epistemic state s, i.e. it should consistently specify truth values of each proposition.

- Such an s can be built directly, cf. the *Ignorant Agent* scenario.
- An appropriate Kripke-style model is also a possible specification device: we build a scaffolding Kripke model $\mathcal{K}_s$ in which s is a possible world.

However, when building $\mathcal{K}_s$, we have to be ready to add to the model (sometimes excessively many) extraneous worlds just to emulate ignorance. Furthermore, a scaffolding Kripke model may lack an epistemic meaning which limits its epistemic value.

Findings and suggestions

The principal contributions of this paper are conceptual.

1. When epistemic logic starts from a given set of possible worlds and defines an accessibility relation in the standard way, the result might not be a Kripke model: only those structures that are fully observable are Kripke models.

The fully observable property is a propositional equivalent to common knowledge of the model, which is quite restrictive but has been tacitly assumed in epistemic modeling. This should be acknowledged.

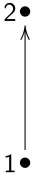
2. We have sketched a basic theory of epistemic models in a general setting without the common knowledge of the model constraint.

This is just a step towards a general theory of epistemic modeling with possible worlds; more expressive tools are needed to capture partial and asymmetric knowledge of the model for multiple agents.

Findings and suggestions

3. A fresh view of intuitionistic semantics.

A well-known tension between intuitionistic Kripke models and the intended constructive semantics of intuitionistic logic can be mitigated by taking into account common knowledge of the model assumption.



Assume that p does not hold in 1 and 2. In the Kripke model $\mathcal{K}$, the agent observes both worlds and this yields a sufficient justification for $\neg p$, hence $1 \Vdash \neg p$ by the Brouwer's witness semantics and Kripkean possible worlds semantics.