On modal logics of trees

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Logics

Intuitionistic (propositional) formulas are in the standard language with a countable set of propositional letters PL and the connectives $\vee, \wedge, \rightarrow, \bot$.

A superintuitionistic (propositional) logic is a set of intuitionistic formulas containing the standard intuitionistic axioms and closed under the rules

- (MP) $A, A \rightarrow B \,/\, A$;
- (Sub) $A \,/\, SA$, where S is a propositional substitution.

Kripke semantics

An intuitionstic Kripke frame is a poset $F = (W, \leq)$.

A Kripke model over F is a pair

- $M = (\Phi, \theta)$, where $\theta : PL \longrightarrow 2^W$ is an intuitionistic valuation, i.e., for each q,
- $u \in \theta(q)$ & $u \leq v \Rightarrow v \in \theta(q)$.

The inductive definition of the truth of an intuitionistic formula A at a point u of a model M ($M, u \Vdash A$) is standard.

A formula A is valid on a frame F ($F \Vdash A$) if $M, u \Vdash A$ for every point u of every model M over F.

$\mathbf{IL}(F) := \{A \mid F \Vdash A\}$ is the superintuitionistic logic of a frame F.

$\mathbf{IL}(\mathcal{C}) := \bigcap\{\mathbf{IL}(F) \mid F \in \mathcal{C}\}$ is the superintuitionistic logic of a class of frames $\mathcal{C}$, or the superintuitionistic logic determined by $\mathcal{C}$.

Completeness and FMP

Logics of the form $\mathbf{IL}(\mathcal{C})$ (or, equivalently, $\mathbf{IL}(F)$)) are called (Kripke) complete.

Logics of the form $\mathbf{IL}(\mathcal{C})$, where $\mathcal{C}$ is a class of finite frames, are said to have the finite model property (FMP).

Every finitely axiomatizable logic with the FMP is decidable.

Intuitionistic trees

DEFINITION (by J. Drugush)
A tree is a poset $(W, \leq)$ with the following properties

- for any u, the set $\{v \mid v \leq u\}$ is a chain,
- $(W, \leq)$ is a lower semilattice.

REMARK. This is a rather broad definition. In particular, every chain is a tree in this sense.

A forest is a disjoint union of trees. A forest logic is a logic determined by a forest.

Theorem (Drugush, 1984) Every superintuitionistic forest logic has the FMP, moreover, it is determined by a forest consisting of finite trees.

Modal formulas and logics

Modal formulas are build from the set PL of proposition letters using the connectives $\to, \bot, \Box$. Other connectives ($\wedge, \vee, \Diamond, \top$ etc.) are abbreviations.

The modal depth $d(A)$ of a modal formula A is defined by induction.

- $d(q) = 0$ for $q \in PL$, $d(\bot) = 0$,
- $d(A \to B) = \max(d(A), d(B))$,
- $d(\Box A) = d(A) + 1$.

A modal logic is a set of modal formulas containing

- the classical tautologies;
- the axiom of **K**: $\Box(p_1 \to p_2) \to (\Box p_1 \to \Box p_2)$,

and closed under the rules

- (MP) $A, A \to B \,/\, A$;
- (Nec) $A \,/\, \Box A$;
- (Sub) $A \,/\, SA$, where S is a propositional substitution.

The minimal modal logic is **K**.

Kripke semantics

An Kripke frame is a non-empty set with a binary relation $F = (W, R)$.

A Kripke model over F is a pair $M = (\Phi, \theta)$, where $\theta : PL \longrightarrow 2^W$ is an valuation.

The inductive definition of the truth of a modal formula A at a point u of a model M ($M, u \vDash A$) is standard.

A formula A is valid on a frame F ($F \vDash A$) if $M, u \vDash A$ for every point u of every model M over F.

$\mathbf{L}(F) := \{A \mid F \vDash A\}$ is the modal logic of a frame F.

$\mathbf{L}(\mathcal{C}) := \bigcap\{\mathbf{L}(F) \mid F \in \mathcal{C}\}$ is the modal logic of a class of frames $\mathcal{C}$, or the modal logic determined by $\mathcal{C}$.

Completeness, FMP, tabularity, local tabularity

Logics of the form $\mathbf{L}(\mathcal{C})$ (or, equivalently, $\mathbf{L}(F)$)) are called (Kripke) complete.

Logics of the form $\mathbf{L}(\mathcal{C})$, where $\mathcal{C}$ is a class of finite frames, are said to have the finite model property (FMP).

FACT 1 Every finitely axiomatizable logic with the FMP is decidable.

A modal logic L is

- locally tabular if for any n there exists finitely many formulas in proposition letters $p_1, \ldots, p_n$, up to equivalence in L ($L \vdash A \leftrightarrow B$).
- tabular if it is determined by a single finite frame.

FACT 2 Every tabular logic is locally tabular.

FACT 3 Every locally tabular logic has the FMP.

FACT 4 Every extension of a locally tabular logic is locally tabular.

Boxing-1

Definition

For a set of modal formulas Γ, put

$$\Box\Gamma := \{\Box A \mid A \in \Gamma\}.$$

If $L = \mathbf{K} + \Gamma$ for a set of formulas Γ, put

$$\Box{\cdot}L := \mathbf{K} + \Box\Gamma.$$

Lemma

$\mathbf{K} + \Gamma \vdash A$ *implies* $\mathbf{K} + \Box\Gamma \vdash \Box A$.

So $\mathbf{K} + \Gamma = \mathbf{K} + \Delta$ implies $\mathbf{K} + \Box\Gamma = \mathbf{K} + \Box\Delta$, i.e., $\Box{\cdot}L$ is well-defined.

It turns out that $\Box{\cdot}L$ inherits many properties of L.

Boxing-2

Theorem 1

- If L is Kripke complete, then $\Box{\cdot}L$ is Kripke complete.
- If L has the FMP, then $\Box{\cdot}L$ has the FMP.
- If L is locally tabular, then $\Box{\cdot}L$ is locally tabular.

Since the logic $\mathbf{Triv} := \mathbf{K} + (p \leftrightarrow \Box p)$ is tabular (it is determined by a single reflexive point), we obtain many examples of locally tabular logics:

Corollary

The logics $\mathbf{K} + \Box^n(p \leftrightarrow \Box p)$ (and all their extensions) are locally tabular.

Trees

Definition

A tree is a rooted frame, in which every point, but the root, has a single predecessor. I.e., this is a frame (W, R) with a point u such that

- $W = \bigcup_{n \geq 0} R^n(u)$,
- $\forall x \neq u \, \exists! y \; yRx$.

A reflexive tree is a reflexive closure of a tree. Similarly we define transitive trees, symmetric trees, etc.

This can be done for any first-order condition on frames expressed by a Horn sentence.

Reflexive trees

Theorem 2

A modal logic determined by any class of reflexive trees has the FMP.

Proof

(Sketch.) It suffices to consider $\mathbf{L}(F)$ for a single reflexive tree F.
Suppose $F, x \not\models A$ for some point x. Let $G = F{\uparrow}x$ be the subtree of F starting at x, and let $n = d(A)$. Consider its truncation $G^{(n)} = G|R^n(x)$. Then $G^{(n)}, x \not\models A$; this is proved for example, by playing a bisimulation game between a countermodel M for A in G and the truncated model $M^{(n)}$, so M, x and $M^{(n)}, x$ are n-bisimilar.
Thus every formula refuted in F is refuted in some $G^{(n)}$.
On the other hand, there is a p-morphism $f : G \twoheadrightarrow G^{(n)}$ such that

- $f(u) = u$ for $u \in G^{(n)}$,
- $f(u) = v$ for $xR^n vR^m u$ (there are unique such v and m).

Reflexive trees (continued)

Proof

(Continued) Thus

$$\mathbf{L}(F) \subseteq \mathbf{L}(G) \subseteq \mathbf{L}(G^{(n)}).$$

It follows that

$$\mathbf{L}(F) = \mathbf{L}(\{(F{\uparrow}x)^{(n)} \mid x \in F,\ n \geq 1\}).$$

Finally observe that every logic $\mathbf{L}((F{\uparrow}x)^{(n)})$ is locally tabular. This follows from a Theorem in

V.B. Shehtman. Bisimulation games and locally tabular logics. Russian Math. Surveys, 71:5 (2016), 979-981.

It states that every logic axiomatized by Chagrov's formula (forbidding paths of different points of length $> n$) is locally tabular.

Therefore, $\mathbf{L}(F)$ has the FMP.

Serial trees

A serial frame is a frame without endpoints (where $R(x) = \varnothing$).
Equivalently, F is serial iff $F \vDash \Diamond\top$.

Theorem 3

A modal logic determined by any class of serial trees has the FMP.

Proof

(Sketch.) Again it suffices to consider $\mathbf{L}(F)$ for a single serial tree F.
The method is almost the same as in Theorem 2. Now we take a truncation $G^{(n)}$ and make all its endpoints reflexive. This gives us a serial frame $G^{(n)\bullet}$, and still $G^{(n)\bullet}, x \nvDash A$.
Since G is serial, we have the same p-morphism $f : G \twoheadrightarrow G^{(n)\bullet}$.
Finally note that $G^{(n)\bullet} \vDash \Box^n(p \leftrightarrow \Box p)$, so by Theorem 1, $\mathbf{L}(G^{(n)\bullet})$ is locally tabular, and we can apply the same argument as in Theorem 2.

More examples of FMP

Theorem 4

The logic of every class of trees validating

$$\Diamond\top \to \Diamond^2\top \wedge \Diamond\Box\bot$$

has the FMP.

Theorem 5

A modal logic determined by any class of reflexive symmetric trees has the FMP.

Counterexample

Theorem 6

There exist a countable tree F such that $\mathbf{L}(F)$ lacks the FMP.

Proof

(Sketch, joint with A. Alexeev). Consider the formula $Alt_1 := \Diamond p \to \Box p$. It is well-known that $(W, R) \models Alt_1$ iff every x has at most one successor.
Let

$$L_0 := \mathbf{K} + \Box Alt_1 + \{Alt_1 \vee (\Diamond^n \Box \bot \to \Diamond^{n+1} \Box \bot) \mid n \geq 1\}.$$

Then

- Every finite frame validating L_0 validates $Alt_1 \vee \Box\Diamond\top$.
- $L_0 \not\vdash Alt_1 \vee \Box\Diamond\top$: there is a corresponding infinite tree F.

So L_0, as well as $\mathbf{L}(F)$, lacks the FMP.

Some questions

1. Does there exist a transitive tree F validating **GL** such that $\mathbf{L}(F)$ lacks the FMP? (Conjecture: yes).
2. Does the modal logic of every reflexive transitive tree enjoy the FMP?
3. Do there exist continuum many superintuitionistic forest logics?
4. Do there exist undecidable logics of trees? (Conjecture: yes).