# Geometry on Words

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## Intended Audience

B.Sc students or M.Sc students in mathematics.

## Course Description

Many interesting examples of words (sequences of letters) are connected with number theory and have explicit geometric interpretation, based on some "self-similarity". For example, Christoffel words have connections with continued fractions. The Tribonacci word 121312112131... is a fixed point of the substitution $1 \to 12$, $2 \to 13$, $3 \to 1$ and encodes a fractal subdivision and a rotation of a 2-dimensional torus. We will investigate these (and not only these) examples and will try to understand connections between combinatorics, number theory, geometry and dynamical systems.

In many questions, the geometric approach gives an intuitive visualization that leads to a better understanding of a problem and sometimes even to its solution. Then, we give an interpretation of the elements of continued fractions in terms of integer geometry, with the continued fractions being associated to certain invariants of integer angles. The geometric viewpoint on continued fractions also gives key ideas for generalizing Gauss–Kuzmin statistics to studying multidimensional Gauss's reduction theory, leading to several results in toric geometry.

The problem of the stability of the Solar system has the property that "it is easy to write out n equations of motion of particles, but it is difficult to understand this motion intuitively." One of the classic examples of dynamical systems is the outer billiard outside the regular square; in particular, it is associated with the problems of the existence of an aperiodic trajectory, as well as the completeness of periodic points. These problems are solved only for a limited number of special cases.

The topics covered in the class will include (but are not limited to):

## Course Outline

- **Rauzy Fractals:** 1) Substitution sequences, examples and basic properties 2) Pisot numbers 3) Definition of Rauzy fractals and examples of fractals 4) Hausdorff distance, contracting maps and equations on sets 5) Connection with rotation of torus, Pisot conjecture
- **Integer Geometry:** 1) Definition of a continued fraction and their properties 2) Geometric continued fractions and their properties 3) Integer angle, triangle and trigonometry and Pick's Formula 4) Gauss's reduction theory and linear algebraic methods
- **Christoffel words:** 1) Combinatorics on Christoffel words 2) Geometric aspects of Christoffel words 3) Continued fractions and Christoffel words 4) Markov numbers and their properties 5) Farey sequence
- **Outer Billiard:**

## Research Talks

- **Alexei Belov:** Words obtained from values of polynomials in integer points
- **Alexei Semenov: TBA**

## Prerequisites/Corequisites

All necessary prerequisites are covered.

## Main References

[1 ] M. Lothaire, Applied Combinatorics on Words, Encyclopedia of Mathematics and its Applications Book 105, Cambridge University, 2005.

[2 ] N. Pytheas Fogg, Substitutions in Dynamics, Arithmetics, and Combinatorics, Lecture Notes in Mathematics, Vol. 1794, Springer, 2002.

[3 ] S. Tabachnikov, Geometry and Billiards, Student Mathematical Library, American Mathematical Society, 2005.

[4 ] O. Karpenkov, Geometry of Continued Fractions, Algorithms and Computation in Mathematics 26, Springer, 2013.

[5 ] J. Berste, Combinatorics on Words: Christoffel Words and Repetitions in Words, CRM Monograph, American Mathematical Society, 2008.

## Grading Policy

The grade will count the assessments using the following proportions:

- **40%** of your grade will be determined by 4 series hometasks (10% each).
- **60%** of your grade will be determined by 1 projects.

**Hometasks:**

Exercises consist of solved problems and research problems based on the ability of the students in the class.