Amer. Math. Soc. Transl.
(2) Vol. **00**, XXXX

Lambek Calculus and Formal Grammars

Mati Pentus

Contents

Introduction

The question about the position of categorial grammars in the Chomsky hierarchy arose in late 1950s and early 1960s. In 1960 Bar-Hillel, Gaifman, and Shamir [**1**] proved that a formal language can be generated by some basic categorial grammar if and only if the language is context-free. They conjectured (see also [**7**]) that the same holds for Lambek grammars, i. e., for categorial grammars based on a syntactic calculus introduced in 1958 by J. Lambek [**10**] (this calculus operates with three connectives: multiplication or concatenation of languages, left division, and right division).

The proof of one half of this conjecture (namely, that every context-free language can be generated by some Lambek grammar) in fact coincides with the proof

1991 *Mathematics Subject Classification.* Primary 68S05; Secondary 03B65.
Key words and phrases. Lambek calculus, categorial grammar, context-free grammar, interpolation property.
The research described in this publication was made possible in part by the Russian Foundation for Basic Research (projects 96-01-01395 and 98-01-00249).

for the case of basic categorial grammars. The converse remained an open problem for several years. A proof was proposed in [**8**], but it contains an error (this was pointed out in [**3**]). W. Buszkowski [**3, 4, 5**] obtained partial results for the fragment without one division and for a product-free fragment with a restriction on division nesting.

In [**2**] J. van Benthem mentioned the conjecture as an open problem of contemporary mathematical linguistics.

From the logical point of view the Lambek calculus is more interesting than the calculus behind basic categorial grammars. In particular, the rule of equivalent type substitution is admissible in the Lambek calculus.

It is known that the Lambek calculus can be embedded into certain fragments of noncommutative linear logic and cyclic linear logic.

Our main aim is to prove the conjecture about context-freeness of all languages generated by Lambek grammars.

This is achieved using a free group interpretation of noncommutative linear logic, a modification of the Craig interpolation property proof by Maehara and Schütte, and combinatorial techniques.

Main results are the following.

(1) We prove context-freeness of languages generated by categorial grammars based on any of the following calculi:
   - the Lambek calculus,
   - the Lambek calculus allowing empty premises,
   - the Lambek calculus with the unit,
   - the multiplicative fragment of cyclic linear logic.

(2) We prove that all elementary fragments of the Lambek calculus have the Craig interpolation property.

(3) We prove that the conjoinability relation (on syntactic types) is decidable and that it is complete with respect to the free group interpretation.

Section 1 defines main notions. Context-free grammars and languages are defined in 1.1.

Subsections 1.2, 1.3, and 1.4 contain definitions related to the Lambek syntactic calculus. This calculus deals with *syntactic types* (we shall call them simply types for shortness) which are built from *primitive types* using three binary connectives: multiplication, left division, and right division.

Lambek categorial grammars, which are based on the Lambek syntactic calculus, are defined in 1.5.

In **Section 2** the free group interpretation of the Lambek calculus is studied. In 2.1 we define this interpretation as the natural translation of the three connectives of the Lambek calculus into multiplication, left division, and right division in a free group.

In 2.2 correctness of the Lambek calculus with respect to this interpretation is proved. Note that completeness with respect to this interpretation does not hold. The exact relation between Lambek calculus derivability and equality of images of types in the free group will be elucidated later in Section 8.

In 2.3 we establish a fact about free groups. This fact will be needed to prove the main lemma in 5.3.

Section 3 introduces the notion of a thin sequent (a sequent in which every primitive type involved in it occurs precisely once positively and once negatively). It is proved that every Lambek calculus derivation can be obtained via substitution from a derivation containing only thin sequents (the same holds for multiplicative fragments of linear logic systems, both commutative and noncommutative).

Section 4 contains the proof of the Craig interpolation theorem for the Lambek calculus (4.1). This proof, based on the technique of Maehara and Schütte, is a simple modification of D. Roorda's proof for a variant of the Lambek calculus allowing empty premises.

In 4.2 we prove that in the case of a thin sequent the length of the interpolant constructed according to the technique of Maehara and Schütte is equal to the length of the reduced word that represents the interpolated part of the original sequent in the free group interpretation. This fact will play essential role in 5.3.

Section 5 is devoted to the proof of the main result: all languages generated by Lambek grammars are context-free. In 5.1 the construction of a context-free grammar corresponding to a given Lambek grammar is given. A finite set of Lambek calculus types is used as the non-terminal alphabet of the context-free grammar. The context-free productions are based on derivable Lambek calculus sequents of bounded length.

The trivial natural relation between context-free grammars and calculi based on the cut rule is formalized in 5.2.

In 5.3 we prove the main lemma, which states that the Lambek calculus is conservative over a calculus corresponding to the context-free grammar constructed in 5.1.

The theorem about context-freeness of all languages generated by Lambek grammars is proved in 5.4. (We give an improved exposition of the proof published in [**12, 14, 16**].)

Section 6 deals with the Craig interpolation property in elementary fragments of the Lambek calculus. We prove that the fragments $L(\backslash, /)$, $L(\backslash)$, and $L(/)$ have the interpolation property (6.1). The same about other elementary fragments is known due to D. Roorda [**18, 19**].

In addition, we introduce the notion of generalized interpolation property, which is of interest in fragments without multiplication. It is proved that the fragments $L(\backslash)$ and $L(/)$ have the generalized interpolation property (6.3), whereas $L(\backslash, /)$ (the product-free Lambek calculus) does not (6.2). These results were first published in [**16**].

In **Section 7** an analog of the main theorem from Section 5 is proved for the product-free Lambek calculus. Here the non-terminal alphabet of the obtained context-free grammar is a finite set of product-free types. The proof is essentially the same as in [**17**]. In [**6**] W. Buszkowski presented a similar proof for the case if the designated type of a Lambek grammar is primitive.

In **Section 8** we give the definition of *conjoinable* syntactic types (from [**2**]) and prove that two types are conjoinable if and only if their free group interpretations are equal (this result was published in [**11, 13, 15**]). This yields a positive answer to the decidability problem for the conjoinability relation. (The problem was formulated in [**2**].)

Section 9 deals with the multiplicative fragment of cyclic linear logic. Here all results and proofs are analogous to those concerning the Lambek calculus.

The multiplicative fragment of cyclic linear logic is defined in 9.1. Next, in 9.2 we define grammars based on this fragment. Correctness of the multiplicative cyclic linear logic with respect to the free group interpretation is established in 9.3. Thin sequents for the multiplicative cyclic linear logic are defined in 9.4. The interpolation theorem for this fragment is proved in 9.5. Finally, in 9.6 we formulate the following theorem: the class of languages generated by grammars based on the multiplicative fragment of the cyclic linear logic coincides with the class of all context-free languages.

The author would like to express his most sincere gratitude to his thesis advisor Prof. S. N. Artemov for formulating the task and for invaluable assistance throughout the research and manuscript preparation, to Prof. M. I. Kanovich for helpful discussions at his lectures on formal grammars in 1992/1993 at Moscow State University, to Prof. V. A. Uspensky, Prof. S. I. Adian, and Prof. J. van Benthem for their attention to this work.

1. Preliminaries

By $\mathbb{N}$ we denote the set of all natural numbers including 0. By $\mathbb{Z}$ we denote the set of all integers.

Let $\mathcal{M}$ be any non-empty set, called an *alphabet*. We shall call its elements *letters*. We define a *word* over the alphabet $\mathcal{M}$ as a finite (possibly empty) sequence $t_1t_2\dots t_n$ of elements of $\mathcal{M}$. Two words $t_1t_2\dots t_n$ and $s_1s_2\dots s_m$ are equal if and only if they coincide as sequences, i. e., if $n=m$ and $t_1=s_1$, $t_2=s_2$, …, $t_n=s_n$. The empty word will be denoted by ε. Let $\mathcal{M}^*$ stand for the set of all words over the alphabet $\mathcal{M}$. The set of all non-empty words over the alphabet $\mathcal{M}$ will be denoted by $\mathcal{M}^+$. We call a *language* any set of words.

The *length* of a word is defined in the natural way: $|t_1t_2\dots t_n| \rightleftharpoons n$.

1.1. Context-free grammars.

Definition 1.1. A *context-free grammar* is a quadruple $\langle\mathcal{T},\mathcal{W},\sigma,\mathcal{R}\rangle$, where $\mathcal{T}$ and $\mathcal{W}$ are two disjoint finite sets, σ is an element of $\mathcal{W}$, and $\mathcal{R}$ is a finite set of *context-free productions* of the form $\alpha\Rightarrow u$, where $\alpha\in\mathcal{W}$ and $u\in(\mathcal{T}\cup\mathcal{W})^+$. The set $\mathcal{T}$ is called the *alphabet of terminal symbols*, whereas the set $\mathcal{W}$ is called the *alphabet of non-terminal symbols*. The symbol σ is called the *start symbol*.

A word w' is *directly derivable* from a word w in a grammar $\langle\mathcal{T},\mathcal{W},\sigma,\mathcal{R}\rangle$ iff $w=v_1\alpha v_2$, $w'=v_1uv_2$ for some $v_1,v_2\in(\mathcal{T}\cup\mathcal{W})^+$, and $\alpha\Rightarrow u$ is a rule from $\mathcal{R}$. We say that w' is *derivable* from w in $\langle\mathcal{T},\mathcal{W},\sigma,\mathcal{R}\rangle$ iff there exists a sequence of words w_0, w_1, … w_n such that $w_i\in(\mathcal{T}\cup\mathcal{W})^*$, $w=w_0$, $w'=w_n$, and for every $i\le n-1$ the word w_{i+1} is directly derivable from w_i. The *language generated by the context-free grammar* $\langle\mathcal{T},\mathcal{W},\sigma,\mathcal{R}\rangle$ (denoted by $\mathcal{G}(\mathcal{T},\mathcal{W},\sigma,\mathcal{R})$), is defined as the set of all words over the alphabet $\mathcal{T}$ that are derivable in this grammar from the one-letter word σ.

Remark 1.2. Many authors allow to use productions of the form $\alpha\Rightarrow\varepsilon$ in context-free grammars. It is well-known that the difference is inessential. Namely, for every context-free grammar that involves rules of the form $\alpha\Rightarrow\varepsilon$ one can effectively construct a context-free grammar in our sense so that the difference of the languages generated by these two grammars is either empty or contains only the empty word (cf. [**9**]).

DEFINITION 1.3. A language is called *context-free* (or *algebraic*) iff there exists a context-free grammar that generates the given language.

1.2. Lambek calculus. We consider the syntactic calculus introduced in [**10**]. We shall denote it by L and call it the *Lambek calculus*. This calculus occupies a central position in modern research in categorial grammar (cf. [**2**, p. 31]).

Assume that a countable set $\mathrm{Var} = \{p_1, p_2, p_3, \ldots\}$ is given. The elements of this set will be referred to as *primitive types*. The Lambek calculus involves three binary connectives $\bullet, \backslash, /$ that are called *multiplication*, *left division*, and *right division*, respectively.

Let Tp be the smallest set satisfying the following two conditions:

- $\mathrm{Var} \subseteq \mathrm{Tp}$;
- if $A \in \mathrm{Tp}$ and $B \in \mathrm{Tp}$, then $(A{\bullet}B) \in \mathrm{Tp}$, $(A\backslash B) \in \mathrm{Tp}$, and $(A/B) \in \mathrm{Tp}$.

The elements of Tp will be called *syntactic types* or simply *types*.

In some cases we shall omit parentheses with the convention that

- $A{\bullet}B{\bullet}C$ stands for $(A{\bullet}B){\bullet}C$;
- the connective $\bullet$ has higher precedence than $\backslash$ and $/$.

Capital letters A, B, ... range over types. Capital Greek letters range over finite (possibly empty) sequences of types. The empty sequence of types is denoted by Λ. The letters p and q range over primitive types.

Sequents of the Lambek calculus are of the form $\Gamma \to A$, where A is a type and Γ is a non-empty sequence of types. The left-hand side of a sequent is called *antecedent* and the right-hand side is called *succedent*.

The axioms of the Lambek calculus are all sequents of the form $p_i \to p_i$, where $p_i \in \mathrm{Var}$.

The derivation rules of the Lambek calculus are the following:

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma\Delta \to A{\bullet}B}\ (\to \bullet), \qquad \frac{\Gamma A B \Delta \to C}{\Gamma(A{\bullet}B)\Delta \to C}\ (\bullet \to),$$

$$\frac{A\Pi \to B}{\Pi \to A\backslash B}\ (\to \backslash),\ \text{where } \Pi \neq \Lambda, \qquad \frac{\Pi \to A \quad \Gamma B \Delta \to C}{\Gamma\Pi(A\backslash B)\Delta \to C}\ (\backslash \to),$$

$$\frac{\Pi A \to B}{\Pi \to B/A}\ (\to /),\ \text{where } \Pi \neq \Lambda, \qquad \frac{\Pi \to A \quad \Gamma B \Delta \to C}{\Gamma(B/A)\Pi\Delta \to C}\ (/ \to),$$

$$\frac{\Pi \to B \quad \Gamma B \Delta \to A}{\Gamma\Pi\Delta \to A}\ (\text{cut}).$$

The cut-elimination theorem for this calculus is proved in [**10**].

We write $\mathcal{C} \vdash \Gamma \to A$ if the sequent $\Gamma \to A$ is derivable in the calculus $\mathcal{C}$. In particular, $L \vdash \Gamma \to A$ means that the sequent $\Gamma \to A$ is derivable in the Lambek calculus.

1.3. Auxiliary notions.

DEFINITION 1.4. The *length* $\|A\|$ of a type A is defined as the total number of primitive type occurrences in A.

$$\|p_i\| \rightleftharpoons 1 \qquad \|A{\bullet}B\| = \|A\backslash B\| = \|A/B\| \rightleftharpoons \|A\| + \|B\|$$

The length of a sequence of types is defined in the natural way.

$$\|A_1 \ldots A_n\| \rightleftharpoons \|A_1\| + \ldots + \|A_n\|$$

Definition 1.5. The set of all primitive types occurring in a type A is denoted by $\mathrm{Var}(A)$.

Definition 1.6. For every primitive type $p \in \mathrm{Var}$ we define two functions $\#_p^+$ and $\#_p^-$ from the set Tp into $\mathbb{N}$, and also a function $\#_p$ from Tp into $\mathbb{Z}$:

$$\begin{array}{rcl}
\#_p^+(q) & \rightleftharpoons & \begin{cases} 1, & \text{if } p = q, \\ 0, & \text{if } q \in \mathrm{Var} \text{ and } p \neq q, \end{cases} \\
\#_p^-(q) & \rightleftharpoons & 0, \text{if } q \in \mathrm{Var}, \\
\#_p^+(A\bullet B) & \rightleftharpoons & \#_p^+(A) + \#_p^+(B), \\
\#_p^-(A\bullet B) & \rightleftharpoons & \#_p^-(A) + \#_p^-(B), \\
\#_p^+(A\backslash B) & \rightleftharpoons & \#_p^-(A) + \#_p^+(B), \\
\#_p^-(A\backslash B) & \rightleftharpoons & \#_p^+(A) + \#_p^-(B), \\
\#_p^+(A/B) & \rightleftharpoons & \#_p^+(A) + \#_p^-(B), \\
\#_p^-(A/B) & \rightleftharpoons & \#_p^-(A) + \#_p^+(B), \\
\\
\#_p(A) & \rightleftharpoons & \#_p^+(A) - \#_p^-(A).
\end{array}$$

These definitions are extended to sequences of types and to sequents as follows:

$$\begin{array}{rcl}
\#_p^+(A_1 \dots A_n) & \rightleftharpoons & \#_p^+(A_1) + \dots + \#_p^+(A_n), \\
\#_p^-(A_1 \dots A_n) & \rightleftharpoons & \#_p^-(A_1) + \dots + \#_p^-(A_n), \\
\#_p(A_1 \dots A_n) & \rightleftharpoons & \#_p(A_1) + \dots + \#_p(A_n),
\end{array}$$

$$\begin{array}{rcl}
\#_p^+(\Pi \to A) & \rightleftharpoons & \#_p^-(\Pi) + \#_p^+(A), \\
\#_p^-(\Pi \to A) & \rightleftharpoons & \#_p^+(\Pi) + \#_p^-(A), \\
\#_p(\Pi \to A) & \rightleftharpoons & \#_p(A) - \#_p(\Pi).
\end{array}$$

1.4. Variants of the Lambek calculus. Let a signature $\Sigma \subseteq \{\backslash, /, \bullet\}$ be given. We denote by $\mathrm{Tp}(\Sigma)$ the set of all types containing only connectives from the given signature Σ.

The *elementary fragment* of the calculus L corresponding to a signature Σ is the calculus obtained from L by removing all types that do not belong to $\mathrm{Tp}(\Sigma)$. We denote this elementary fragment by $L(\Sigma)$.

The calculus $L(\backslash, /)$ is called the *product-free Lambek calculus*.

Definition 1.7. The calculus L^* (cf. [**2**]) is obtained from the original Lambek calculus L by allowing antecedents to be empty and dropping the condition $\Pi \neq \Lambda$ in the rules $(\to \backslash)$ and $(\to /)$.

Next we define the calculus $L^*_{\mathbf{1}}$ (the Lambek calculus with the unit).

Definition 1.8. Let $\mathrm{Tp}_{\mathbf{1}}$ be the smallest set satisfying the following conditions:

- $\mathbf{1} \in \mathrm{Tp}_{\mathbf{1}}$;
- $\mathrm{Var} \subseteq \mathrm{Tp}_{\mathbf{1}}$;
- if $A \in \mathrm{Tp}_{\mathbf{1}}$ and $B \in \mathrm{Tp}_{\mathbf{1}}$, then $(A\bullet B) \in \mathrm{Tp}_{\mathbf{1}}$, $(A\backslash B) \in \mathrm{Tp}_{\mathbf{1}}$, and $(A/B) \in \mathrm{Tp}_{\mathbf{1}}$.

The sequents of the calculus $L^*_{\mathbf{1}}$ are of the form $\Gamma \to A$, where $A \in \mathrm{Tp}_{\mathbf{1}}$ and $\Gamma \in (\mathrm{Tp}_{\mathbf{1}})^*$.

The axioms of $L^*_{\mathbf{1}}$ are all sequents of the form $p_i \to p_i$, where $p_i \in \mathrm{Var}$, as well as the sequent $\to \mathbf{1}$. The calculus $L^*_{\mathbf{1}}$ has all the derivation rules of L^* and, in addition, the rule $(\mathbf{1} \to)$:

$$\frac{\Gamma\Delta \to C}{\Gamma\mathbf{1}\Delta \to C}\ (\mathbf{1} \to).$$

1.5. Lambek grammars.

Definition 1.9. A *Lambek grammar* is a triple $\langle \mathcal{T}, H, \triangleright \rangle$, where $\mathcal{T}$ is a finite set (the alphabet), H is a type of the Lambek calculus, and $\triangleright$ is a finite binary relation $\triangleright \subset \mathrm{Tp} \times \mathcal{T}$.

The *language generated by the Lambek grammar* $\langle \mathcal{T}, H, \triangleright \rangle$ is defined as the set of all strings $t_1 \dots t_n$ over the alphabet $\mathcal{T}$ for which there exists a derivable (in L) sequent $B_1 \dots B_n \to H$ such that $B_i \triangleright t_i$ for all $i \le n$. We shall denote this language by $\mathcal{L}_L(\mathcal{T}, H, \triangleright)$.

Categorial grammars based on other sequent calculi are defined similarly. For the sake of unification of definitions we stipulate that the empty word is not included in the languages generated by grammars based on L^* and $L^*_{\mathbf{1}}$.

2. Free group interpretation

2.1. Definition of free group interpretation. Let $F(\mathrm{Var})$ stand for the free group generated by the enumerable set of all primitive types $\mathrm{Var} = \{p_1, p_2, p_3, \dots\}$. By *free group* we mean the following particular representation.

We introduce the extended alphabet Var', obtained by adding to the set Var a new symbol p_i^{-1} for each $p_i \in \mathrm{Var}$. We shall consider reduced words over this extended alphabet. A word u over the alphabet Var' is said to be *reduced* if it does not contain adjacent occurrences of p_i and p_i^{-1}. The empty word, denoted by ε, is also reduced. The set $F(\mathrm{Var})$ consists of all reduced words. Multiplication on this set is defined by induction on word length.

- If $u = u'p_i$ and $v = p_i^{-1}v'$ for some i, then $uv \rightleftharpoons u'v'$.
- If $u = u'p_i^{-1}$ and $v = p_iv'$ for some i, then $uv \rightleftharpoons u'v'$.
- Otherwise uv is obtained simply by juxtaposition.

It is obvious that the product of any two reduced words is reduced. The identity element of the free group $F(\mathrm{Var})$ is the empty word ε.

For any element $u \in F(\mathrm{Var})$, we define $|u|$ as the length of the reduced word u.

Definition 2.1. The *free group interpretation* (written as $[\![\ ]\!]$) is the following mapping of types and finite sequences of types into $F(\mathrm{Var})$.

$$\begin{aligned} [\![p_i]\!] &\rightleftharpoons p_i \\ [\![A\bullet B]\!] &\rightleftharpoons [\![A]\!][\![B]\!] \\ [\![A\backslash B]\!] &\rightleftharpoons [\![A]\!]^{-1}[\![B]\!] \\ [\![A/B]\!] &\rightleftharpoons [\![A]\!][\![B]\!]^{-1} \end{aligned}$$

$$[\![A_1 \dots A_n]\!] \rightleftharpoons [\![A_1]\!] \dots [\![A_n]\!]$$

Lemma 2.2. *For any type A, $|[\![A]\!]| \le \|A\|$.*

Proof. By induction on the construction of A. □

2.2. Soundness.

LEMMA 2.3. *If a sequent* $\Gamma \to C$ *is derivable in the Lambek calculus, then* $[\![\Gamma]\!] = [\![C]\!]$.

D. Roorda obtained this result in terms of "atomic markings" and "balance". We present an immediate proof (from [**11**]) in terms of free groups.

PROOF. Induction on derivations.
Case 1. Axiom.
Trivial.
Case 2. $(\to \bullet)$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma\Delta \to A\bullet B}\ (\to \bullet)$$

By the induction hypothesis $[\![\Gamma]\!] = [\![A]\!]$ and $[\![\Delta]\!] = [\![B]\!]$. Consequently $[\![\Gamma\Delta]\!] = [\![A]\!][\![B]\!] = [\![A\bullet B]\!]$.
Case 3. $(\bullet \to)$

$$\frac{\Gamma A B \Delta \to C}{\Gamma(A\bullet B)\Delta \to C}\ (\bullet \to)$$

Obvious.
Case 4. $(\to \backslash)$

$$\frac{A\Pi \to B}{\Pi \to A\backslash B}\ (\to \backslash)$$

Multiplying the equality $[\![A]\!][\![\Pi]\!] = [\![B]\!]$ by $[\![A]\!]^{-1}$ on the left, one obtains $[\![\Pi]\!] = [\![A]\!]^{-1}[\![B]\!]$. Thus $[\![\Pi]\!] = [\![A\backslash B]\!]$.
Case 5. $(\to /)$
Similar to the previous case.
Case 6. $(\backslash \to)$

$$\frac{\Pi \to A \quad \Gamma B \Delta \to C}{\Gamma\Pi(A\backslash B)\Delta \to C}\ (\backslash \to)$$

If $[\![\Pi]\!] = [\![A]\!]$, then $[\![\Pi]\!][\![A]\!]^{-1} = \varepsilon$. On the other hand, $[\![\Gamma]\!][\![B]\!][\![\Delta]\!] = [\![C]\!]$. Thus $[\![\Gamma]\!][\![\Pi]\!][\![A]\!]^{-1}[\![B]\!][\![\Delta]\!] = [\![C]\!]$, which yields $[\![\Gamma]\!][\![\Pi]\!][\![A\backslash B]\!][\![\Delta]\!] = [\![C]\!]$.
Case 7. $(/ \to)$
Similar to the previous case. □

2.3. A property of free groups. The following lemma demonstrates that juxtaposed reduced words can reduce to the empty word only if at least one of the given words "loses" at least half of its symbols during reduction with one of its immediate neighbors.

LEMMA 2.4. *If* $u_1, \ldots, u_n \in F(\mathrm{Var})$, $n > 1$, *and* $u_1 \ldots u_n = \varepsilon$, *then there exists an index* $k < n$ *such that* $|u_k u_{k+1}| \le \max(|u_k|, |u_{k+1}|)$.

PROOF. For any two elements u_i and u_{i+1} in $F(\mathrm{Var})$ there exist three reduced words x_i, $y_{i,i+1}$, and z_{i+1} in $F(\mathrm{Var})$ such that $u_i = x_i y_{i,i+1}$, $u_{i+1} = y_{i,i+1}^{-1} z_{i+1}$, $u_i u_{i+1} = x_i z_{i+1}$, and the words $x_i y_{i,i+1}$, $y_{i,i+1}^{-1} z_{i+1}$, $x_i z_{i+1}$ are reduced. Evidently, $|u_i| = |x_i| + |y_{i,i+1}|$, $|u_{i+1}| = |y_{i,i+1}^{-1}| + |z_{i+1}| = |y_{i,i+1}| + |z_{i+1}|$ and $|u_i u_{i+1}| = |x_i| + |z_{i+1}|$.

Assume for the contrary that the inequalities $|u_i u_{i+1}| > |u_i|$ and $|u_i u_{i+1}| > |u_{i+1}|$ hold for every index $i < n$. From $|u_i u_{i+1}| > |u_i|$ we obtain $|x_i| + |y_{i,i+1}| < |x_i| + |z_{i+1}|$, whence $|y_{i,i+1}| < |z_{i+1}|$, and consequently $|y_{i,i+1}| <$

$\frac{1}{2}|u_{i+1}|$. Similarly, from $|u_i u_{i+1}| > |u_{i+1}|$ we derive $|y_{i,i+1}| + |z_{i+1}| < |x_i| + |z_{i+1}|$, whence $|y_{i,i+1}| < |x_i|$, which in turn yields $|y_{i,i+1}| < \frac{1}{2}|u_i|$.

Now let us consider an arbitrary index i such that $1 < i < n$. Recall that $u_i = y_{i-1,i}^{-1} z_i$ and, on the other hand, $u_i = x_i y_{i,i+1}$. Both words $y_{i-1,i}^{-1} z_i$ and $x_i y_{i,i+1}$ are reduced and thus they coincide. In view of $|y_{i-1,i}^{-1}| < \frac{1}{2}|u_i|$ and $|y_{i,i+1}| < \frac{1}{2}|u_i|$, we have $u_i = y_{i-1,i}^{-1} w_i y_{i,i+1}$, $x_i = y_{i-1,i}^{-1} w_i$, and $z_i = w_i y_{i,i+1}$ for a suitable reduced word w_i. Note that both $y_{i-1,i}^{-1} w_i$ and $w_i y_{i,i+1}$ are reduced.

Substituting in $u_1 \dots u_n = \varepsilon$ the word $x_1 y_{1,2}$ for u_1, $y_{n-1,n}^{-1} z_n$ for u_n, and $y_{i-1,i}^{-1} w_i y_{i,i+1}$ for u_i (where $1 < i < n$), we obtain $x_1 w_2 w_3 \dots w_{n-1} z_n = \varepsilon$.

Now we can check that the word $x_1 w_2 w_3 \dots w_{n-1} z_n$ is reduced. Note that $x_1 w_2$ is reduced, since $x_1 z_2 = x_1(w_2 y_{2,3})$ is reduced. Similarly, $w_{n-1} z_n$ is reduced, since $x_{n-1} z_n = (y_{n-2,n-1}^{-1} w_{n-1}) z_n$ is reduced. Finally, for every index i satisfying $1 < i < n-1$ the word $w_i w_{i+1}$ is reduced, for $x_i z_{i+1} = (y_{i-1,i}^{-1} w_i)(w_{i+1} y_{i+1,i+2})$ is reduced.

Thus we have established that the word $x_1 w_2 w_3 \dots w_{n-1} z_n$ is reduced and $x_1 w_2 w_3 \dots w_{n-1} z_n = \varepsilon$. Consequently each of the words x_1, w_2, w_3, $\dots$, w_{n-1}, and z_n is empty. But they all must be non-empty, because $|y_{i-1,i}^{-1}| < \frac{1}{2}|u_i|$ and $|y_{i,i+1}| < \frac{1}{2}|u_i|$. Contradiction. □

3. Thin sequents

In this section we introduce the notion of "thin" sequents and show that every sequent derivable in the Lambek calculus may be obtained from some thin sequent via substitution.

Definition 3.1. A sequent $\Pi \to A$ is *thin* iff the inequalities $\#_p^+(\Pi \to A) \le 1$ and $\#_p^-(\Pi \to A) \le 1$ hold for every $p \in \mathrm{Var}$ (i. e., each primitive type occurs in the sequent positively at most once and negatively at most once).

Definition 3.2. We shall refer to any function from Var to Var as a *primitive type substitution.*

Every primitive type substitution ϕ induces a function from Tp to Tp (also denoted by ϕ):

$$\begin{aligned} \phi(E\bullet F) &\rightleftharpoons \phi(E)\bullet\phi(F); \\ \phi(E\backslash F) &\rightleftharpoons \phi(E)\backslash\phi(F); \\ \phi(E/F) &\rightleftharpoons \phi(E)/\phi(F). \end{aligned}$$

We also extend the function ϕ to sequents:

$$\phi(E_1 \dots E_m \to F) \rightleftharpoons \phi(E_1) \dots \phi(E_m) \to \phi(F).$$

Lemma 3.3. *Let ϕ be a primitive type substitution. If we replace in any Lambek calculus derivation every sequent $\Gamma \to C$ by $\phi(\Gamma \to C)$, then the tree obtained is a correct derivation in the Lambek calculus.*

Proof. Induction on derivation length. □

Theorem 3.4. *A sequent $\Pi \to A$ is derivable in the Lambek calculus if and only if there exist a thin derivable sequent $\Theta \to B$ and a primitive type substitution ϕ such that $\Pi \to A = \phi(\Theta \to B)$.*

EXAMPLE 3.5. Consider the sequent $(p/p)\,p \to p/(p\backslash p)$ in the role of $\Pi \to A$. This sequent can be derived in the Lambek calculus as follows:

$$\dfrac{p \to p \qquad \dfrac{p \to p \quad p \to p}{(p/p)\,p \to p}\,(/\to)}{\dfrac{(p/p)\,p\,(p\backslash p) \to p}{(p/p)\,p \to p/(p\backslash p)}\,(\to /).}\,(\backslash\to)$$

Take $(q_3/q_2)\,q_1 \to q_3/(q_1\backslash q_2)$ as $\Theta \to B$. Let $\phi(q_1) = p$, $\phi(q_2) = p$, $\phi(q_3) = p$. Then

$$\dfrac{q_1 \to q_1 \qquad \dfrac{q_2 \to q_2 \quad q_3 \to q_3}{(q_3/q_2)\,q_2 \to q_3}\,(/\to)}{\dfrac{(q_3/q_2)\,q_1\,(q_1\backslash q_2) \to q_3}{(q_3/q_2)\,q_1 \to q_3/(q_1\backslash q_2)}\,(\to /).}\,(\backslash\to)$$

PROOF OF THEOREM 3.4. The 'if' part is an immediate consequence of Lemma 3.3.

To prove the 'only if' part we consider an arbitrary cut-free derivation of $\Pi \to A$. Let n be the number of axiom instances in the derivation. (Evidently, $\|\Pi\| + \|A\| = 2n$.) We introduce n new primitive types $q_1, \ldots, q_n$ and assume a one-to-one correspondence between axiom instances and new primitive types being given. The substitution ϕ is defined as follows. If a new primitive type q_i corresponds to an axiom instance $p_j \to p_j$, then $\phi(q_i) \rightleftharpoons p_j$.

Now we turn the given cut-free derivation of $\Pi \to A$ into a derivation of $\Gamma \to C$ so that $\phi(\Gamma \to C) = \Pi \to A$ and the derivation structure remains the same. First we replace each axiom instance by an axiom instance containing the corresponding new primitive type. Next we spread this replacement down along the derivation tree. This is possible due to the fact that in all derivation rules except the cut rule every primitive type occurrence in the consequence has exactly one predecessor in the premises of the rule. □

4. Interpolation

In 1991 D. Roorda [**18**] proved (using the method of Maehara and Schütte [**20**]) that the calculus L^* has the Craig interpolation property. In the paper [**19**] he remarked that the proof handles also the case of L. In Section 4.1 we present a proof of the interpolation theorem for L. Essentially this proof copies D. Roorda's proof for L^*.

The interpolation property in elementary fragments of the Lambek calculus will be studied in Section 6.

4.1. Interpolation in $L(\backslash, /, \bullet)$.

LEMMA 4.1. *Let* $L \vdash \Phi\Theta\Psi \to C$, *where* $\Phi \in \mathrm{Tp}^*$, $\Theta \in \mathrm{Tp}^+$, $\Psi \in \mathrm{Tp}^*$, *and* $C \in \mathrm{Tp}$. *Then there exists a type* E *such that*

(i) $L \vdash \Theta \to E$;
(ii) $L \vdash \Phi E \Psi \to C$;
(iii) *the inequality* $\#_p^+(E) \le \min(\#_p^+(\Theta), \#_p^+(C) + \#_p^-(\Phi\Psi))$ *holds for every primitive type* p;
(iv) *the inequality* $\#_p^-(E) \le \min(\#_p^-(\Theta), \#_p^-(C) + \#_p^+(\Phi\Psi))$ *holds for every primitive type* p.

We shall write $\Phi[\Theta]\Psi \to C$ instead of $\Phi\Theta\Psi \to C$ in order to show the selected part of the antecedent.

Every type E that satisfies clauses (i) – (iv) is referred to as an *interpolant* for Θ in the sequent $\Phi[\Theta]\Psi \to C$.

Proof of Lemma 4.1. Induction on the length of a cut-free derivation.

Case 1. Let $\Phi\Theta\Psi \to C$ be an axiom, i. e., $C = \Phi\Theta\Psi$. From $\Theta \in \mathrm{Tp}^+$ it follows that $\Theta = C$ and $\Phi = \Psi = \Lambda$. We put $E = C$.

In all the following cases the given partition of the conclusion of a rule induces partitions of premises. By induction hypothesis one can find interpolants for the premises.

Case 2. Consider the rule $(\to \backslash)$.

$$\frac{A\Phi[\Theta]\Psi \to B}{\Phi[\Theta]\Psi \to A\backslash B}\ (\to \backslash)$$

By the induction hypothesis there is a type E such that $L \vdash \Theta \to E$, $L \vdash A\Phi E\Psi \to B$, $\#_p^+(E) \le \min(\#_p^+(\Theta), \#_p^+(B) + \#_p^-(A\Phi\Psi))$, and $\#_p^-(E) \le \min(\#_p^-(\Theta), \#_p^-(B) + \#_p^+(A\Phi\Psi))$ for every $p \in \mathrm{Var}$.

We verify that (i), (ii), (iii), and (iv) hold for the conclusion of the rule $(\to \backslash)$ with the same interpolant E as for the premise. The clause (i) is evident from the induction hypothesis. The derivation

$$\frac{A\Phi E\Psi \to B}{\Phi E\Psi \to A\backslash B}\ (\to \backslash)$$

establishes (ii). The clauses (iii) and (iv) follow from the induction hypothesis and the definition of $\#_p^+$ and $\#_p^-$.

Case 3. The rule $(\to /)$ is handled similarly.

Case 4. For the rule $(\backslash \to)$ we consider six subcases.

Case 4a.

$$\frac{\Pi'[\Pi'']\Pi''' \to A \quad \Gamma B\Delta \to C}{\Gamma\Pi'[\Pi'']\Pi'''(A\backslash B)\Delta \to C}\ (\backslash \to).$$

Similar to case 2.

Case 4b.

$$\frac{\Pi \to A \quad \Gamma'[\Gamma'']\Gamma''' B\Delta \to C}{\Gamma'[\Gamma'']\Gamma'''\Pi(A\backslash B)\Delta \to C}\ (\backslash \to).$$

Similar to case 2.

Case 4c.

$$\frac{\Pi \to A \quad \Gamma B\Delta'[\Delta'']\Delta''' \to C}{\Gamma\Pi(A\backslash B)\Delta'[\Delta'']\Delta''' \to C}\ (\backslash \to).$$

Similar to case 2.

Case 4d.

$$\frac{[\Pi']\Pi'' \to A \quad \Gamma'[\Gamma'']B\Delta \to C}{\Gamma'[\Gamma''\Pi']\Pi''(A\backslash B)\Delta \to C}\ (\backslash \to).$$

Let E be an interpolant for the left premise and F be an interpolant for the right premise. It is easy to verify that $F{\bullet}E$ is an interpolant for the conclusion of the rule $(\backslash \to)$. The clause (i) is proved by the derivation

$$\frac{\Gamma'' \to F \quad \Pi' \to E}{\Gamma''\Pi' \to F{\bullet}E}\ ({\bullet} \to).$$

Case 4e.

$$\frac{\Pi \to A \quad \Gamma'[\Gamma'' B \Delta']\Delta'' \to C}{\Gamma'[\Gamma''\Pi(A\backslash B)\Delta']\Delta'' \to C}\ (\backslash \to).$$

Let E be an interpolant for the right premise. We prove that E is also an interpolant for the conclusion. The proof of (ii) is obvious. Using the induction hypothesis (i) we obtain

$$\frac{\Pi \to A \quad \Gamma'' B \Delta' \to E}{\Gamma''\Pi(A\backslash B)\Delta' \to E}\ (\backslash \to).$$

This establishes (i). To prove (iii) observe that $\#_p^+(B) + \#_p^-(\Gamma''\Delta') \leq \#_p^+(A\backslash B) + \#_p^-(\Gamma''\Pi\Delta')$. The clause (iv) is verified similarly.

Case 4f.

$$\frac{[\Pi']\Pi'' \to A \quad \Gamma[B\Delta']\Delta'' \to C}{\Gamma\Pi'[\Pi''(A\backslash B)\Delta']\Delta'' \to C}\ (\backslash \to).$$

Let E be an interpolant for the right premise and F be an interpolant for the left premise. We show that the type $F\backslash E$ is an interpolant for the conclusion. First we verify (iii):

$$\begin{aligned}
&\#_p^+(F\backslash E) = \\
&\quad \#_p^+(E) + \#_p^-(F) \leq \\
&\quad \min(\#_p^+(B\Delta'), \#_p^+(C) + \#_p^-(\Gamma\Delta'')) + \min(\#_p^-(A) + \#_p^+(\Pi''), \#_p^-(\Pi')) \leq \\
&\quad \min(\#_p^+(B\Delta') + \#_p^-(A) + \#_p^+(\Pi''), \#_p^+(C) + \#_p^-(\Gamma\Delta'') + \#_p^-(\Pi')) = \\
&\quad \min(\#_p^+(\Pi''(A\backslash B)\Delta'), \#_p^+(C) + \#_p^-(\Gamma\Pi'\Delta'')).
\end{aligned}$$

Next we derive (i):

$$\cfrac{\cfrac{F\Pi'' \to A \quad B\Delta' \to E}{F\Pi''(A\backslash B)\Delta' \to E}\ (\backslash \to)}{\Pi''(A\backslash B)\Delta' \to F\backslash E}\ (\to \backslash).$$

Finally, (ii) is proved by

$$\frac{\Pi' \to F \quad \Gamma E \Delta'' \to C}{\Gamma\Pi'(F\backslash E)\Delta'' \to C}\ (\backslash \to).$$

Case 5. The rule $(/ \to)$ is handled similarly to case 4.
Case 6. For the rule $(\to \bullet)$ three subcases arise.
Case 6a.

$$\frac{\Gamma'[\Gamma'']\Gamma''' \to A \quad \Delta \to B}{\Gamma'[\Gamma'']\Gamma'''\Delta \to A\bullet B}\ (\to \bullet).$$

Similar to case 2.
Case 6b.

$$\frac{\Gamma'[\Gamma''] \to A \quad [\Delta']\Delta'' \to B}{\Gamma'[\Gamma''\Delta']\Delta'' \to A\bullet B}\ (\to \bullet).$$

Similar to case 4d.
Case 6c.

$$\frac{\Gamma \to A \quad \Delta'[\Delta'']\Delta''' \to B}{\Gamma\Delta'[\Delta'']\Delta''' \to A\bullet B}\ (\to \bullet).$$

Similar to case 2.
Case 7. For the rule $(\bullet \to)$ we consider three subcases, all of which are handled similarly to case 2.

Case 7a.

$$\frac{\Gamma'[\Gamma'']\Gamma'''AB\Delta \to C}{\Gamma'[\Gamma'']\Gamma'''(A\bullet B)\Delta \to C}\ (\bullet \to).$$

Case 7b.

$$\frac{\Gamma'[\Gamma''AB\Delta']\Delta'' \to C}{\Gamma'[\Gamma''(A\bullet B)\Delta']\Delta'' \to C}\ (\bullet \to).$$

Case 7c.

$$\frac{\Gamma AB\Delta'[\Delta'']\Delta''' \to C}{\Gamma(A\bullet B)\Delta'[\Delta'']\Delta''' \to C}\ (\bullet \to).$$

□

REMARK 4.2. The clauses (iii) and (iv) imply that $\#_p^-(E) + \#_p^+(E) \le \min(\#_p^-(\Theta) + \#_p^+(\Theta), \#_p^-(\Phi\Psi C) + \#_p^+(\Phi\Psi C))$.

COROLLARY 4.3. *Let $L \vdash A \to C$, where $A \in \mathrm{Tp}$ and $C \in \mathrm{Tp}$. Then there is a type E such that*

(i) $L \vdash A \to E$;
(ii) $L \vdash E \to C$;
(iii) $\mathrm{Var}(E) \subseteq \mathrm{Var}(A) \cap \mathrm{Var}(C)$.

PROOF. We apply Lemma 4.1 with $\Phi = \Lambda$, $\Theta = A$, and $\Psi = \Lambda$. Consider an arbitrary primitive type p that occurs in the interpolant E. Remark 4.2 shows that $\min(\#_p^- A + \#_p^+ A, \#_p^- C + \#_p^+ C) \ge \#_p^- E + \#_p^+ E \ge 1$, whence $\#_p^- A + \#_p^+ A \ge 1$ and $\#_p^- C + \#_p^+ C \ge 1$, i. e., $p \in \mathrm{Var}(A)$ and $p \in \mathrm{Var}(C)$. □

EXAMPLE 4.4. Consider the derivable sequent $p_1\bullet(p_1\backslash p_2) \to (p_3/p_2)\backslash p_3$. Here $\mathrm{Var}(p_1\bullet(p_1\backslash p_2)) \cap \mathrm{Var}((p_3/p_2)\backslash p_3) = \{p_1, p_2\} \cap \{p_2, p_3\} = \{p_2\}$.

The type $E = p_2$ is an interpolant for this sequent. Really, $L \vdash p_1\bullet(p_1\backslash p_2) \to p_2$ and $L \vdash p_2 \to (p_3/p_2)\backslash p_3$.

4.2. Interpolation property for thin sequents.

LEMMA 4.5. *Let $L \vdash \Phi\Theta\Psi \to C$, where $\Phi \in \mathrm{Tp}^*$, $\Theta \in \mathrm{Tp}^+$, $\Psi \in \mathrm{Tp}^*$, $C \in \mathrm{Tp}$, and the sequent $\Phi\Theta\Psi \to C$ is thin. Then there is a type E such that*

(i) $L \vdash \Theta \to E$;
(ii) $L \vdash \Phi E\Psi \to C$;
(iii) *the sequent* $\Theta \to E$ *is thin;*
(iv) *the sequent* $\Phi E \Psi \to C$ *is thin;*
(v) $\|E\| = |[\![\Theta]\!]|$.

PROOF. According to Lemma 4.1 there is a type E satisfying (i) and (ii). It remains to prove (iii), (iv), and (v).

Consider an arbitrary primitive type p. In view of $\#_p^+(E) \le \#_p^+(C) + \#_p^-(\Phi\Psi)$, we have $\#_p^+(E) + \#_p^-(\Theta) \le \#_p^+(C) + \#_p^-(\Phi\Theta\Psi) \le 1$ (the last inequality follows from the original sequent $\Phi\Theta\Psi \to C$ being thin).

Similarly, $\#_p^-(E) + \#_p^+(\Theta) \le \#_p^-(C) + \#_p^+(\Phi\Theta\Psi) \le 1$. This proves (iii).

The clause (iv) can be verified in an analogous manner.

To establish (v) it is sufficient to verify that $\|E\| = |[\![E]\!]|$. This is obvious, since no primitive type occurs in E more than once. □

5. Main theorem

5.1. Construction.

Definition 5.1. For every natural number m we define a set of bounded types $\mathrm{Tp}(m)$ and a set of bounded type sequences $\mathrm{Ls}(m)$:

$$\begin{array}{rcl}\mathrm{Tp}(m) & \rightleftharpoons & \{A \in \mathrm{Tp} \mid \|A\| \leq m\};\\ \mathrm{Ls}(m) & \rightleftharpoons & \{\Pi \in \mathrm{Tp}(m)^+ \mid \|\Pi\| \leq 2m\}.\end{array}$$

Definition 5.2. For every pair of natural numbers m and s we define a finite set of types $\mathrm{Tp}(m,s)$ and a finite set of type sequences $\mathrm{Ls}(m,s)$:

$$\begin{array}{rcl}\mathrm{Tp}(m,s) & \rightleftharpoons & \{A \in \mathrm{Tp} \mid \mathrm{Var}(A) \subseteq \{p_1,\ldots,p_s\} \text{ and } \|A\| \leq m\};\\ \mathrm{Ls}(m,s) & \rightleftharpoons & \{\Pi \in \mathrm{Tp}(m,s)^+ \mid \|\Pi\| \leq 2m\}.\end{array}$$

Consider an arbitrary Lambek grammar $\langle \mathcal{T}, H, \triangleright\rangle$. Only a finite number of types are relevant in the definition of the language generated by this Lambek grammar. Thus there are positive integers m and s such that $H \in \mathrm{Tp}(m,s)$ and if $B \triangleright t$ for some $t \in \mathcal{T}$, then $B \in \mathrm{Tp}(m,s)$.

There is no loss of generality in assuming that the sets $\mathcal{T}$ and $\mathrm{Tp}(m,s)$ do not intersect. Now we construct the desired context-free grammar $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R}\rangle$:

$$\begin{array}{rcl}\mathcal{W} & \rightleftharpoons & \mathrm{Tp}(m,s),\\ \sigma & \rightleftharpoons & H,\\ \mathcal{R} & \rightleftharpoons & \{B \Rightarrow t \;\mid\; t \in \mathcal{T} \text{ and } B \triangleright t\} \cup\\ & & \quad \{A \Rightarrow \Gamma \mid A \in \mathrm{Tp}(m,s),\ \Gamma \in \mathrm{Ls}(m,s), \text{ and } L \vdash \Gamma \to A\}.\end{array}$$

The aim of this section is to prove that $\mathcal{L}_L(\mathcal{T}, H, \triangleright) = \mathcal{G}(\mathcal{T}, \mathcal{W}, \sigma, \mathcal{R})$.

Lemma 5.3. *Let $t_1, \ldots, t_n \in \mathcal{T}$. Then the word $t_1 \ldots t_n$ is in $\mathcal{G}(\mathcal{T}, \mathcal{W}, \sigma, \mathcal{R})$ if and only if there are symbols $\alpha_1, \ldots, \alpha_n \in \mathcal{W}$ such that the word $\alpha_1 \ldots \alpha_n$ is derivable from σ, and $(\alpha_i \Rightarrow t_i) \in \mathcal{R}$ for every $i \leq n$.*

Proof. Observe that every derivation of $\Rightarrow t_1 \ldots t_n$ from σ in the constructed context-free grammar $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R}\rangle$ can be reorganized so that all occurrences of productions $B \Rightarrow t$, where $t \in \mathcal{T}$, appear after all occurrences of productions $A \Rightarrow \Gamma$, where $\Gamma \in \mathrm{Ls}(m,s)$. □

5.2. Calculus representation of context-free grammars.

Definition 5.4. Given a context-free grammar $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R}\rangle$ we construct a calculus $C_1(\mathcal{W}, \sigma, \mathcal{R})$, derivable objects of which are sequents of the form $w \to \sigma$, where $w \in \mathcal{W}^+$.

- The only axiom of $C_1(\mathcal{W}, \sigma, \mathcal{R})$ is $\sigma \to \sigma$.
- If $(\alpha \Rightarrow u) \in \mathcal{R}$, $\alpha \in \mathcal{W}$, and $u \in \mathcal{W}^+$, then the calculus $C_1(\mathcal{W}, \sigma, \mathcal{R})$ contains the rule

$$\frac{v_1 \alpha v_2 \to \sigma}{v_1 u v_2 \to \sigma.}$$

Lemma 5.5. *Let $w \in \mathcal{W}^+$. The sequent $w \to \sigma$ is derivable in the calculus $C_1(\mathcal{W}, \sigma, \mathcal{R})$ if and only if the word w is derivable from σ in the context-free grammar $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R}\rangle$.*

Proof. The 'if' part is proved by induction on derivation length in the context-free grammar $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R} \rangle$.

Induction base: $w = \sigma$. Obvious.

Induction step: let $w = v_1 u v_2$, $(\alpha \Rightarrow u) \in \mathcal{R}$, and $v_1 \alpha v_2$ be derivable from σ in $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R} \rangle$. We apply the rule

$$\frac{v_1 \alpha v_2 \to \sigma}{v_1 u v_2 \to \sigma.}$$

The 'only if' part is proved similarly by induction on derivation length in the calculus $C_1(\mathcal{W}, \sigma, \mathcal{R})$. □

Definition 5.6. Given a context-free grammar $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R} \rangle$ we construct a calculus $C_2(\mathcal{W}, \mathcal{R})$, derivable objects of which are sequents of the form $w \to \alpha$, where $w \in \mathcal{W}^+$, and $\alpha \in \mathcal{W}$.

- The calculus $C_2(\mathcal{W}, \mathcal{R})$ contains an axiom $\alpha \to \alpha$ for every symbol α.
- If $(\alpha \Rightarrow u) \in \mathcal{R}$, $\alpha \in \mathcal{W}$, and $u \in \mathcal{W}^+$, then the calculus $C_2(\mathcal{W}, \mathcal{R})$ contains the axiom $u \to \alpha$.
- The only rule of the calculus $C_2(\mathcal{W}, \mathcal{R})$ is the cut rule

$$\frac{u \to \alpha \quad v_1 \alpha v_2 \to \beta}{v_1 u v_2 \to \beta.}$$

Lemma 5.7. *Let $w \in \mathcal{W}^+$. A sequent $w \to \sigma$ is derivable in the calculus $C_2(\mathcal{W}, \mathcal{R})$ if and only if the word w is derivable from σ in the context-free grammar $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R} \rangle$.*

Proof. In view of Lemma 5.5, it suffices to prove that a sequent $w \to \sigma$ is derivable in the calculus $C_2(\mathcal{W}, \mathcal{R})$ if and only if it is derivable in the calculus $C_1(\mathcal{W}, \sigma, \mathcal{R})$.

The 'only if' part is easy to verify. To prove the 'if' part we define the rank of a cut as the number of sequents in the derivation of its left premise and proceed by induction on the total of ranks of all cuts in a given derivation.

A derivation fragment

$$\cfrac{\cfrac{u \to \alpha \quad v_1 \alpha v_2 \to \beta}{v_1 u v_2 \to \beta} \qquad w_1 \beta w_2 \to \gamma}{w_1 v_1 u v_2 w_2 \to \gamma}$$

will be replaced by

$$\cfrac{u \to \alpha \qquad \cfrac{v_1 \alpha v_2 \to \beta \quad w_1 \beta w_2 \to \gamma}{w_1 v_1 \alpha v_2 w_2 \to \gamma}}{w_1 v_1 u v_2 w_2 \to \gamma.}$$

□

5.3. Main lemma. In this section we establish a correspondence between the Lambek calculus and the calculus $C_2(\mathcal{W}, \mathcal{R})$ representing the context-free grammar constructed in Section 5.1.

For every natural number m we introduce an auxiliary calculus $Lcut_m$, which in some sense takes the intermediate position between the calculi L and $C_2(\mathcal{W}, \mathcal{R})$. Namely, the calculus $C_2(\mathcal{W}, \mathcal{R})$ uses formulas from $\mathrm{Tp}(m, s)$, the calculus $Lcut_m$ uses $\mathrm{Tp}(m)$, and L uses formulas from Tp.

DEFINITION 5.8. A sequent $\Gamma \to A$ is an axiom of $Lcut_m$ iff $A \in \mathrm{Tp}(m)$, $\Gamma \in \mathrm{Ls}(m)$, and the sequent $\Gamma \to A$ is derivable in the Lambek calculus. The only rule of $Lcut_m$ is (cut).

LEMMA 5.9. *Let* $L \vdash \Pi \to C$, *where* $\Pi \in \mathrm{Ls}(m)$, $C \in \mathrm{Tp}(m)$, *and the sequent* $\Pi \to C$ *is thin. Then* $Lcut_m \vdash \Pi \to C$.

PROOF. Induction on $\|\Pi\|$. If $\|\Pi\| \le 2m$, then $\Pi \to C$ is an axiom of $Lcut_m$.

Assume that $\|\Pi\| > 2m$. The sequence Π can be represented as a concatenation $\Pi = \Pi_1 \dots \Pi_l$, where

- $0 < \|\Pi_i\| \le m$ for every $i \le l$,
- $\|\Pi_i\| + \|\Pi_{i+1}\| > m$ for every $i < l$.

Note that $[\![\Pi]\!] = [\![C]\!]$ according to Lemma 2.3. Now let $u_1 \rightleftharpoons [\![\Pi_1]\!], \dots, u_l \rightleftharpoons [\![\Pi_l]\!]$, and $u_{l+1} \rightleftharpoons [\![C]\!]^{-1}$. Evidently $u_1 \dots u_l u_{l+1} = \varepsilon$. Applying Lemma 2.4 we find a positive integer $k \le l$ such that $|u_k u_{k+1}| \le \max(|u_k|, |u_{k+1}|)$. According to Lemma 2.2 the inequality $|u_i| \le m$ holds for every $i \le l+1$. Thus $|u_k u_{k+1}| \le m$.

The following two cases arise.

Case 1. Let $k < l$.

Then $|[\![\Pi_k \Pi_{k+1}]\!]| \le m$ for this particular k. Applying Lemma 4.5 for

$$\underbrace{\Pi_1 \dots \Pi_{k-1}}_{\Phi}\,\underbrace{\Pi_k \Pi_{k+1}}_{\Theta}\,\underbrace{\Pi_{k+2} \dots \Pi_l}_{\Psi} \to C$$

we find an interpolant E for $\Pi_k \Pi_{k+1}$ in $\Pi_1 \dots \Pi_l \to C$. This means that $\|E\| \le m$ and the sequents $\Pi_k \Pi_{k+1} \to E$ and $\Pi_1 \dots \Pi_{k-1} E \Pi_{k+2} \dots \Pi_l \to C$ are thin and derivable.

Note that $\|E\| \le m$, but $\|\Pi_k \Pi_{k+1}\| > m$. Thus $\|\Pi_1 \dots \Pi_{k-1} E \Pi_{k+2} \dots \Pi_l\| < \|\Pi_1 \dots \Pi_l\|$ and we can apply the induction hypothesis for the thin derivable sequent $\Pi_1 \dots \Pi_{k-1} E \Pi_{k+2} \dots \Pi_l \to C$.

On the other hand $\Pi_k \Pi_{k+1} \to E$ is an axiom of $Lcut_m$, since $\|E\| \le m$ and $\|\Pi_k \Pi_{k+1}\| \le 2m$.

Thus we have demonstrated that $Lcut_m \vdash \Pi_1 \dots \Pi_{k-1} E \Pi_{k+2} \dots \Pi_l \to C$ and $Lcut_m \vdash \Pi_k \Pi_{k+1} \to E$. Now $Lcut_m \vdash \Pi_1 \dots \Pi_{k-1} \Pi_k \Pi_{k+1} \Pi_{k+2} \dots \Pi_l \to C$ is obtained by applying the cut rule. We have proved that $Lcut_m \vdash \Pi \to C$.

Case 2. Let $k = l$.

Then $|[\![\Pi_l]\!][\![C]\!]^{-1}| \le m$. Applying Lemma 4.5 for

$$\underbrace{\Pi_1 \dots \Pi_{l-1}}_{\Theta}\,\underbrace{\Pi_l}_{\Psi} \to C$$

we find an interpolant E for $\Pi_1 \dots \Pi_{l-1}$ in $\Pi_1 \dots \Pi_l \to C$. This means that $\|E\| = |[\![\Pi_1 \dots \Pi_{l-1}]\!]|$ and the sequents $\Pi_1 \dots \Pi_{l-1} \to E$ and $E \Pi_l \to C$ are thin and derivable.

Recall that $[\![\Pi_1 \dots \Pi_{l-1} \Pi_l]\!] = [\![C]\!]$, whence $[\![\Pi_1 \dots \Pi_{l-1}]\!] = [\![C]\!][\![\Pi_l]\!]^{-1} = ([\![\Pi_l]\!][\![C]\!]^{-1})^{-1}$ and further, $|[\![\Pi_1 \dots \Pi_{l-1}]\!]| = |([\![\Pi_l]\!][\![C]\!]^{-1})^{-1}| = |[\![\Pi_l]\!][\![C]\!]^{-1}| \le m$. Thus $\|E\| = |[\![\Pi_1 \dots \Pi_{l-1}]\!]| \le m$. It follows that $E \Pi_l \in \mathrm{Ls}(m)$, and consequently $E \Pi_l \to C$ is an axiom of $Lcut_m$.

On the other hand, $\|\Pi_1 \dots \Pi_{l-1}\| < \|\Pi_1 \dots \Pi_l\|$ and we can apply the induction hypothesis for the thin derivable sequent $\Pi_1 \dots \Pi_{l-1} \to E$.

Applying the cut rule we obtain $Lcut_m \vdash \Pi_1 \dots \Pi_{l-1} \Pi_l \to C$. In other words $Lcut_m \vdash \Pi \to C$. □

LEMMA 5.10. *Let $\langle \mathcal{T}, H, \triangleright \rangle$ be a Lambek grammar and $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R} \rangle$ be the corresponding context-free grammar (constructed in Section 5.1). Let $\Gamma \in \mathcal{W}^+$ and $A \in \mathcal{W}$. Then the following three assertions are equivalent.*

(i) $C_2(\mathcal{W}, \mathcal{R}) \vdash \Gamma \to A$
(ii) $Lcut_m \vdash \Gamma \to A$
(iii) $L \vdash \Gamma \to A$

PROOF. The implication (i) $\longrightarrow$ (iii) is easily verified by induction on derivation length in $C_2(\mathcal{W}, \mathcal{R})$.

To prove (ii) $\longrightarrow$ (i) we consider the following primitive type substitution:

$$\phi_s(p_i) \rightleftharpoons \begin{cases} p_i, & \text{if } i \leq s \\ p_1, & \text{if } i > s. \end{cases}$$

Note that ϕ_s maps $\mathrm{Tp}(m)$ to $\mathrm{Tp}(m, s) = \mathcal{W}$. The substitution ϕ_s is applied to all sequents in a derivation of the given sequent $\Gamma \to A$ in $Lcut_m$. The resulting tree is a derivation in $C_2(\mathcal{W}, \mathcal{R})$.

It remains to establish (iii) $\longrightarrow$ (ii). Let $L \vdash \Gamma \to A$, where $\Gamma \in \mathcal{W}^+$. According to Theorem 3.4 there exist a thin derivable sequent $\Pi \to C$ and a primitive type substitution ϕ such that $\Gamma \to A = \phi(\Pi \to C)$. According to Lemma 5.9 we have $Lcut_m \vdash \Pi \to C$. Consequently also the sequent $\Gamma \to A$ is derivable in the calculus $Lcut_m$. $\square$

5.4. Proof of the main theorem.

THEOREM 5.11. *Let $\langle \mathcal{T}, H, \triangleright \rangle$ be a Lambek grammar. Then the language $\mathcal{L}_L(\mathcal{T}, H, \triangleright)$ is context-free.*

PROOF. We prove that $\mathcal{L}_L(\mathcal{T}, H, \triangleright) = \mathcal{G}(\mathcal{T}, \mathcal{W}, \sigma, \mathcal{R})$, where $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R} \rangle$ is the context-free grammar constructed in Section 5.1.

Given a word $t_1 \ldots t_n$, consider the following chain of equivalent assertions.

(1) The word $t_1 \ldots t_n$ is in the language $\mathcal{G}(\mathcal{T}, \mathcal{W}, \sigma, \mathcal{R})$.
(2) There are types $B_1, \ldots, B_n$ such that $B_1 \ldots B_n$ is derivable from σ in $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R} \rangle$, and $(B_i \Rightarrow t_i) \in \mathcal{R}$ for every $i \leq n$.
(3) There are types $B_1, \ldots, B_n$ such that $C_2(\mathcal{W}, \mathcal{R}) \vdash B_1 \ldots B_n \to \sigma$ and $B_i \triangleright t_i$ holds for every $i \leq n$.
(4) There are types $B_1, \ldots, B_n$ such that $L \vdash B_1 \ldots B_n \to \sigma$ and $B_i \triangleright t_i$ holds for every $i \leq n$.
(5) The word $t_1 \ldots t_n$ is in the language $\mathcal{L}_L(\mathcal{T}, H, \triangleright)$.

The equivalence of (1) and (2) is established by Lemma 5.3. Further, (2) and (3) are equivalent according to Lemma 5.7 and the construction of the set $\mathcal{R}$. The equivalence of (3) and (4) follows from Lemma 5.10. Finally, (4) and (5) are equivalent due to the definition of the language generated by a Lambek grammar. $\square$

COROLLARY 5.12. *A language is context-free if and only if it is generated by some Lambek grammar.*

REMARK 5.13. All the arguments above hold also for the Lambek calculus with the unit and for the calculus L^*. Consequently, the class of languages generated by categorial grammars based on any of these calculi coincides with the class of all context-free languages.

6. Interpolation in fragments

In this section we introduce the *generalized interpolation property* and study both ordinary and generalized interpolation in all elementary fragments of the calculi L and L^*. In particular, we prove a "weak interpolation theorem" for the product-free fragment of the Lambek calculus. This is used in the proof of existence of a "natural" context-free grammar for every categorial grammar based on this important fragment of the Lambek calculus.

DEFINITION 6.1. Let $\Sigma \subseteq \{\backslash, /, \bullet\}$ and $\mathcal{C}$ be either $L(\Sigma)$ or $L^*(\Sigma)$. We say that the calculus $\mathcal{C}$ has the *interpolation property* if for every derivable in $\mathcal{C}$ sequent of the form $A \to C$, where $A \in \mathrm{Tp}(\Sigma)$ and $C \in \mathrm{Tp}(\Sigma)$, there exists a type $B \in \mathrm{Tp}(\Sigma)$ such that

(i) $\mathcal{C} \vdash A \to B$;
(ii) $\mathcal{C} \vdash B \to C$;
(iii) $\mathrm{Var}(B) \subseteq \mathrm{Var}(A) \cup \mathrm{Var}(C)$.

DEFINITION 6.2. Let $\Sigma \subseteq \{\backslash, /, \bullet\}$ and $\mathcal{C}$ be either $L(\Sigma)$ or $L^*(\Sigma)$. We say that the calculus $\mathcal{C}$ has the *generalized interpolation property* if for every derivable in $\mathcal{C}$ sequent of the form $\Pi \to C$, where $\Pi \in \mathrm{Tp}(\Sigma)^*$ and $C \in \mathrm{Tp}(\Sigma)$, there exists a type $B \in \mathrm{Tp}(\Sigma)$ such that

(i) $\mathcal{C} \vdash \Pi \to B$;
(ii) $\mathcal{C} \vdash B \to C$;
(iii) $\mathrm{Var}(B) \subseteq \mathrm{Var}(\Pi) \cup \mathrm{Var}(C)$.

REMARK 6.3. Evidently, if multiplication is not in the signature of $\mathcal{C}$, then $\mathcal{C}$ has the generalized interpolation property if and only if it has the ordinary interpolation property (since $\mathcal{C} \vdash A_1 \dots A_n \to C$ if and only if $\mathcal{C} \vdash (A_1 \bullet \dots \bullet A_n) \to C$).

REMARK 6.4. In view of duality of signatures $\{\backslash\}$ and $\{/\}$, as well as duality of $\{\backslash, \bullet\}$ and $\{/, \bullet\}$, it is sufficient to study the fragments based on the signatures $\{\backslash\}$, $\{\backslash, /\}$, $\{\bullet\}$, $\{\backslash, \bullet\}$, and $\{\backslash, /, \bullet\}$.

In 1991 D. Roorda proved that the calculi L^* and $L^*(\backslash, \bullet)$ have the interpolation property [**18, 19**]. The same question for the fragments $L(\backslash)$, $L(\backslash, /)$, $L^*(\backslash)$, and $L^*(\backslash, /)$ was mentioned as an open problem [**19**, p. 440]. Note that the interpolation property for fragments $L^*(\bullet)$, $L(\bullet)$, L, and $L(\backslash, \bullet)$ is easily obtained from the proofs of D. Roorda.

In this section we prove the following results.

- The fragments $L(\backslash)$ and $L^*(\backslash)$ have both ordinary and generalized interpolation property.
- The fragments $L(\backslash, /)$ and $L^*(\backslash, /)$ have the ordinary interpolation property, but do not have the generalized interpolation property.
- The fragments $L(\backslash, /)$ and $L^*(\backslash, /)$ satisfy a certain weak version of the generalized interpolation property.

We give only the proofs for fragments of L, since the proofs for fragments of L^* are analogous.

6.1. "Weak" generalized interpolation.

LEMMA 6.5. *Let $\Phi \in \mathrm{Tp}(\backslash, /)^*$, $\Theta \in \mathrm{Tp}(\backslash, /)^*$, $\Psi \in \mathrm{Tp}(\backslash, /)^*$, $C \in \mathrm{Tp}(\backslash, /)$, and $L \vdash \Phi\Theta\Psi \to C$. Then there is a natural number $r \geq 0$, there are sequences of types $\Theta_1, \dots, \Theta_r \in \mathrm{Tp}(\backslash, /)^+$, and there are types $E_1, \dots, E_r \in \mathrm{Tp}(\backslash, /)$ such that*

(i) $\Theta_1 \dots \Theta_r = \Theta$, *i.e., the sequence* Θ *is divided into* r *non-empty continuous subsequences (if* $\Theta = \Lambda$, *then* $r = 0$*);*
(ii) $L \vdash \Theta_j \to E_j$ *for every* $j \le r$*;*
(iii) $L \vdash \Phi E_1 \dots E_r \Psi \to C$*;*
(iv) $\#_p^+(E_1 \dots E_r) \le \min(\#_p^+(\Theta), \#_p^+(C) + \#_p^-(\Phi\Psi))$ *and* $\#_p^-(E_1 \dots E_r) \le \min(\#_p^-(\Theta), \#_p^-(C) + \#_p^+(\Phi\Psi))$ *for every* $p \in \mathrm{Var}$.

We say that the sequence $E_1 \dots E_r$ is an *interpolant* for Θ in the sequent $\Phi\Theta\Psi \to C$.

Example 6.6. Consider the derivable sequent $[p_1(p_1 \backslash p_2)p_3](p_3 \backslash (p_2 \backslash p_4)) \to p_4$. Applying Lemma 4.1 we obtain a division of the selected subsequence $p_1(p_1 \backslash p_2)p_3 = \Theta_1\Theta_2$, where $\Theta_1 = p_1(p_1 \backslash p_2)$ and $\Theta_2 = p_3$ (here $r = 2$). The corresponding interpolant is $p_2 p_3$, i.e., $E_1 = p_2$ and $E_2 = p_3$. Really, $L \vdash p_1(p_1 \backslash p_2) \to p_2$, $L \vdash p_3 \to p_3$, and $L \vdash p_2 p_3 (p_3 \backslash (p_2 \backslash p_4)) \to p_4$. Note that no single product-free formula is an interpolant for $p_1(p_1 \backslash p_2)p_3$ in $[p_1(p_1 \backslash p_2)p_3](p_3 \backslash (p_2 \backslash p_4)) \to p_4$.

Proof of Lemma 6.5. Induction on the length of a cut-free derivation.
Case 1. Let $\Phi\Theta\Psi \to C$ be an axiom, i.e., $C = \Phi\Theta\Psi$. Three subcases arise from different partitions of the antecedent between Φ, Θ, and Ψ.
Case 1a. Let $[C] \to C$. We put $r = 1$, $\Theta_1 = C$, $E_1 = C$.
Case 1b. Let $[\,]C \to C$. We put $r = 0$.
Case 1c. Let $C[\,] \to C$. We put $r = 0$.

In all the following cases we shall consider the partition of premises induced by the given partition of the conclusion of a rule. By induction hypothesis there exist interpolants for the premises.
Case 2. Consider the rule $(\to \backslash)$

$$\frac{A\Phi[\Theta]\Psi \to B}{\Phi[\Theta]\Psi \to A \backslash B}\ (\to \backslash).$$

By the induction hypothesis we find $\Theta_1, \dots, \Theta_r, E_1, \dots, E_r$ such that $\Theta_1 \dots \Theta_r = \Theta$, $L \vdash \Theta_j \to E_j$ for every $j \le r$, $A\Phi E_1 \dots E_r \Psi \to B$, $\#_p^+(E_1 \dots E_r) \le \min(\#_p^+(\Theta), \#_p^+(B) + \#_p^-(A\Phi\Psi))$ and $\#_p^-(E_1 \dots E_r) \le \min(\#_p^-(\Theta), \#_p^-(B) + \#_p^+(A\Phi\Psi))$ for every $p \in \mathrm{Var}$.

We verify that (i), (ii), (iii), and (iv) hold for the conclusion of the rule $(\to \backslash)$ with the same $\Theta_1, \dots, \Theta_r, E_1, \dots, E_r$ as for the premise. The clauses (i) and (ii) are evident from the induction hypothesis. The derivation

$$\frac{A\Phi E_1 \dots E_r \Psi \to B}{\Phi E_1 \dots E_r \Psi \to A \backslash B}\ (\to \backslash)$$

establishes (iii). The clause (iv) follows from the induction hypothesis and the definition of $\#_p^-$ and $\#_p^+$.
Case 3. The rule $(\to /)$ is treated similarly.
Case 4. Consider the rule $(\backslash \to)$. Six subcases arise.
Case 4a.

$$\frac{\Pi'[\Pi'']\Pi''' \to A \quad \Gamma B \Delta \to C}{\Gamma\Pi'[\Pi'']\Pi'''(A \backslash B)\Delta \to C}\ (\backslash \to)$$

Similar to case 2.
Case 4b.

$$\frac{\Pi \to A \quad \Gamma'[\Gamma'']\Gamma''' B \Delta \to C}{\Gamma'[\Gamma'']\Gamma'''\Pi(A \backslash B)\Delta \to C}\ (\backslash \to)$$

Similar to case 2.

Case 4c.

$$\frac{\Pi \to A \quad \Gamma B \Delta'[\Delta'']\Delta''' \to C}{\Gamma \Pi (A \backslash B) \Delta'[\Delta'']\Delta''' \to C}\ (\backslash \to)$$

Similar to case 2.

Case 4d.

$$\frac{[\Pi']\Pi'' \to A \quad \Gamma'[\Gamma'']B\Delta \to C}{\Gamma'[\Gamma''\Pi']\Pi''(A \backslash B)\Delta \to C}\ (\backslash \to)$$

Let $E_1 \dots E_r$ and $F_1 \dots F_m$ be the interpolants for the left and right premises respectively. It is easy to verify that $F_1 \dots F_m E_1 \dots E_r$ is an interpolant for the conclusion of the rule $(\backslash \to)$.

Case 4e.

$$\frac{\Pi \to A \quad \Gamma'[\Gamma'' B \Delta']\Delta'' \to C}{\Gamma'[\Gamma''\Pi(A \backslash B)\Delta']\Delta'' \to C}\ (\backslash \to)$$

Let $E_1 \dots E_r$ be the interpolant for the right premise. We prove that it is also an interpolant for the conclusion. The clauses (i) and (iii) are obvious.

By the induction hypothesis, $\Gamma'' B \Delta' = \Theta_1 \dots \Theta_r$. Let the particular type occurrence B be in the sequence Θ_k. Then $\Theta_k = \Xi B \Upsilon$ for some sequences Ξ and Υ.

We put $\tilde{\Theta}_k = \Xi \Pi (A \backslash B) \Upsilon$ and $\tilde{\Theta}_j = \Theta_j$ for every $j \neq k$. Evidently $\Gamma''\Pi(A \backslash B)\Delta' = \tilde{\Theta}_1 \dots \tilde{\Theta}_r$. Using the induction hypothesis (ii) we obtain

$$\frac{\Pi \to A \quad \Xi B \Upsilon \to E_k}{\Xi \Pi (A \backslash B) \Upsilon \to E_k}\ (\backslash \to)$$

and $\tilde{\Theta}_j \to E_j$ for every $j \neq k$. This proves (ii). To prove (iv), it is sufficient to observe that $\#_p^+(\Gamma'' B \Delta') \leq \#_p^+(\Gamma''\Pi(A \backslash B)\Delta')$ and $\#_p^-(\Gamma'' B \Delta') \leq \#_p^-(\Gamma''\Pi(A \backslash B)\Delta')$.

Case 4f.

$$\frac{[\Pi']\Pi'' \to A \quad \Gamma[B\Delta']\Delta'' \to C}{\Gamma\Pi'[\Pi''(A \backslash B)\Delta']\Delta'' \to C}\ (\backslash \to)$$

Let $E_1 \dots E_r$ be an interpolant for the right premise, corresponding to the partition $B\Delta' = \Theta_1 \dots \Theta_r$. Let $F_1 \dots F_m$ be an interpolant for the left premise, corresponding to the partition $\Pi' = \Xi_1 \dots \Xi_m$.

Then, for a suitable sequence Υ,

(1) $\Theta_1 = B\Upsilon$;
(2) $\Delta' = \Upsilon \Theta_2 \dots \Theta_r$;
(3) $L \vdash B\Upsilon \to E_1$;
(4) $L \vdash \Theta_j \to E_j$ for every $j \neq 1$;
(5) $L \vdash \Gamma E_1 \dots E_r \Delta'' \to C$;
(6) $\#_p^+(E_1 \dots E_r) \leq \min(\#_p^+(B\Delta'), \#_p^+(C) + \#_p^-(\Gamma\Delta''))$ for every $p \in \mathrm{Var}$;
(7) $\#_p^-(E_1 \dots E_r) \leq \min(\#_p^-(B\Delta'), \#_p^-(C) + \#_p^+(\Gamma\Delta''))$ for every $p \in \mathrm{Var}$;
(8) $\Pi' = \Xi_1 \dots \Xi_m$;
(9) $L \vdash \Xi_j \to F_j$ for every $j \leq m$;
(10) $L \vdash F_1 \dots F_m \Pi'' \to A$;
(11) $\#_p^+(F_1 \dots F_m) \leq \min(\#_p^+(\Pi'), \#_p^+(A) + \#_p^-(\Pi''))$ for every $p \in \mathrm{Var}$;
(12) $\#_p^-(F_1 \dots F_m) \leq \min(\#_p^-(\Pi'), \#_p^-(A) + \#_p^+(\Pi''))$ for every $p \in \mathrm{Var}$.

We show that $(F_m\backslash(\ldots\backslash(F_1\backslash E_1)\ldots))\,E_2\ldots E_r$ is an interpolant for the conclusion, corresponding to the partition $\Pi''(A\backslash B)\Delta' = \tilde{\Theta}_1\ldots\tilde{\Theta}_r$, where $\tilde{\Theta}_1 = \Pi''(A\backslash B)\Upsilon$ and $\tilde{\Theta}_j = \Theta_j$ for every $j \neq 1$.

First, we prove (iv):

$$\begin{aligned}
&\#_p^+((F_m\backslash(\ldots\backslash(F_1\backslash E_1)\ldots))\,E_2\ldots E_r) = \\
&\quad \#_p^+(E_1\ldots E_r) + \#_p^-(F_1\ldots F_m) \leq \\
&\quad \min(\#_p^+(B\Delta'), \#_p^+(C) + \#_p^-(\Gamma\Delta'')) + \min(\#_p^-(A) + \#_p^+(\Pi''), \#_p^-(\Pi')) \leq \\
&\quad \min(\#_p^+(B\Delta') + \#_p^-(A) + \#_p^+(\Pi''), \#_p^+(C) + \#_p^-(\Gamma\Delta'') + \#_p^-(\Pi')) = \\
&\quad \min(\#_p^+(\Pi''(A\backslash B)\Delta'), \#_p^+(C) + \#_p^-(\Gamma\Pi'\Delta'')).
\end{aligned}$$

Evidently, (i) holds, since $\Pi''(A\backslash B)\Delta' = \tilde{\Theta}_1\ldots\tilde{\Theta}_r$.

To prove (ii) we only need to verify that $L \vdash \tilde{\Theta}_1 \to (F_m\backslash(\ldots\backslash(F_1\backslash E_1)\ldots))$. Indeed,

$$\dfrac{\dfrac{\dfrac{\dfrac{F_1\ldots F_m\Pi'' \to A \quad B\Upsilon \to E_1}{F_1\ldots F_m\Pi''(A\backslash B)\Upsilon \to E_1}\,(\backslash\to)}{F_2\ldots F_m\Pi''(A\backslash B)\Upsilon \to F_1\backslash E_1}\,(\to\backslash)}{\vdots}\,(\to\backslash)}{\Pi''(A\backslash B)\Upsilon \to (F_m\backslash(\ldots\backslash(F_1\backslash E_1)\ldots))}\,(\to\backslash).$$

Finally, we prove (iii):

$$\dfrac{\Xi_m \to F_m \quad \dfrac{\dfrac{\dfrac{\Xi_1 \to F_1 \quad \Gamma E_1 E_2\ldots E_r\Delta'' \to C}{\Gamma\Xi_1(F_1\backslash E_1)E_2\ldots E_r\Delta'' \to C}\,(\backslash\to)}{\vdots}\,(\backslash\to)}{\Gamma\Xi_1\ldots\Xi_{m-1}(F_{m-1}\backslash(\ldots\backslash(F_1\backslash E_1)\ldots))E_2\ldots E_r\Delta'' \to C}\,(\backslash\to)}{\Gamma\Xi_1\ldots\Xi_{m-1}\Xi_m(F_m\backslash(F_{m-1}\backslash(\ldots\backslash(F_1\backslash E_1)\ldots)))E_2\ldots E_r\Delta'' \to C}\,(\backslash\to).$$

Case 5. The rule $(/\to)$ is handled similarly to case 4. □

LEMMA 6.7. *Let $\Phi \in \mathrm{Tp}(\backslash)^*$, $\Theta \in \mathrm{Tp}(\backslash)^*$, $\Psi \in \mathrm{Tp}(\backslash)^*$, $C \in \mathrm{Tp}(\backslash)$, and $L \vdash \Phi\Theta\Psi \to C$. Then there is a natural number $r \geq 0$, there are sequences of types $\Theta_1,\ldots,\Theta_r \in \mathrm{Tp}(\backslash)^+$, and there are types $E_1,\ldots,E_r \in \mathrm{Tp}(\backslash)$ such that*

- (i) $\Theta_1\ldots\Theta_r = \Theta$, *i.e., the sequence Θ is divided into r non-empty continuous subsequences (if $\Theta = \Lambda$, then $r = 0$);*
- (ii) $L \vdash \Theta_j \to E_j$ *for every $j \leq r$;*
- (iii) $L \vdash \Phi E_1\ldots E_r\Psi \to C$;
- (iv) *the inequalities* $\#_p^+(E_1\ldots E_r) \leq \min(\#_p^+(\Theta), \#_p^+(C) + \#_p^-(\Phi\Psi))$ *and* $\#_p^-(E_1\ldots E_r) \leq \min(\#_p^-(\Theta), \#_p^-(C) + \#_p^+(\Phi\Psi))$ *hold for every $p \in \mathrm{Var}$.*

PROOF. Induction on the length of a cut-free derivation. It suffices to repeat cases 1, 2, and 4 from the proof of Lemma 6.5. □

6.2. The elementary fragment $\{\backslash, /\}$.

LEMMA 6.8. *The calculus $L(\backslash, /)$ has the interpolation property.*

PROOF. Let $L(\backslash, /) \vdash A \to C$. According to Lemma 6.5 there is an interpolant $E_1\ldots E_r$ for A in the sequent $A \to C$. Obviously $r = 1$. We put $B \rightleftharpoons E_1$. □

LEMMA 6.9. *The calculus $L(\backslash, /)$ does not have the generalized interpolation property.*

PROOF. Consider the sequent $p_1p_2 \to p_3/(p_2\backslash(p_1\backslash p_3))$. It can be derived as follows:

$$\dfrac{p_2 \to p_2 \qquad \dfrac{p_1 \to p_1 \quad p_3 \to p_3}{p_1(p_1\backslash p_3) \to p_3}\,(\backslash\to)}{\dfrac{p_1p_2(p_2\backslash(p_1\backslash p_3)) \to p_3}{p_1p_2 \to p_3/(p_2\backslash(p_1\backslash p_3))}\,(\to/).}\,(\backslash\to)$$

We prove that there is no single-type interpolant for p_1p_2 in this sequent. Assume for the contrary that there exists a type $E \in \mathrm{Tp}(\backslash,/)$ such that $L \vdash p_1p_2 \to E$, $L \vdash E \to p_3/(p_2\backslash(p_1\backslash p_3))$, and $\mathrm{Var}(E) \subseteq \{p_1, p_2\}$.

We need the following translation $(\ )^{cl}$ that maps Lambek calculus types to propositional logic formulas:

$$\begin{array}{rcl} p^{cl} & \rightleftharpoons & p, \text{ if } p \in \mathrm{Var}, \\ (A\backslash B)^{cl} & \rightleftharpoons & A^{cl} \supset B^{cl}, \\ (A/B)^{cl} & \rightleftharpoons & B^{cl} \supset A^{cl}, \\ (A\bullet B)^{cl} & \rightleftharpoons & A^{cl}\& B^{cl}. \end{array}$$

This translation is extended to sequents as follows:

$$(A_1 \ldots A_n \to B)^{cl} \quad \rightleftharpoons \quad (A_1^{cl}\& \ldots \& A_n^{cl}) \supset B^{cl}.$$

It is routine to verify that if $L \vdash \Pi \to C$ then the formula $(\Pi \to C)^{cl}$ is true in the classical propositional logic.

In particular, the formulas $(p_1\&p_2) \supset E^{cl}$ and $E^{cl} \supset ((p_2 \supset (p_1 \supset p_3)) \supset p_3)$ are true. Substituting $\bot$ for p_3 in the latter formula, we obtain $E^{cl} \supset (p_1\&p_2)$. Thus the pure implicative formula E^{cl} is classically equivalent to the formula $(p_1\&p_2)$. Contradiction. □

6.3. The elementary fragment $\{\backslash\}$.

LEMMA 6.10. *Let $\Phi \in \mathrm{Tp}(\backslash)^*$, $\Theta \in \mathrm{Tp}(\backslash)^+$, $C \in \mathrm{Tp}(\backslash)$, and $L \vdash \Phi\Theta \to C$. Then there is a type $E \in \mathrm{Tp}(\backslash)$ such that*

(i) $L \vdash \Theta \to E$;
(ii) $L \vdash \Phi E \to C$;
(iii) *the inequalities* $\#_p^+(E) \leq \min(\#_p^+(\Theta), \#_p^+(C) + \#_p^-(\Phi\Psi))$ *and* $\#_p^-(E) \leq \min(\#_p^-(\Theta), \#_p^-(C) + \#_p^+(\Phi\Psi))$ *hold for every* $p \in \mathrm{Var}$.

PROOF. Induction on the length of a cut-free derivation. All cases, except the following two, are trivial.

Case 1.

$$\frac{\Pi \to A \quad \Gamma'[\Gamma'' B \Delta] \to C}{\Gamma'[\Gamma''\Pi(A\backslash B)\Delta] \to C}\,(\backslash\to)$$

Following case 4e from the proof of Lemma 4.1 it is easy to verify that the interpolant for the right premise is also an interpolant for the conclusion.

Case 2.

$$\frac{[\Pi']\Pi'' \to A \quad \Gamma[B\Delta] \to C}{\Gamma\Pi'[\Pi''(A\backslash B)\Delta] \to C}\,(\backslash\to)$$

Let E be an interpolant for the right premise. Applying Lemma 6.7 to the left premise we obtain an interpolant $F_1 \ldots F_m$. Now, following case 4f from the proof of Lemma 6.5 one can verify that $(F_m\backslash(\ldots\backslash(F_1\backslash E)\ldots))$ is the desired interpolant. □

COROLLARY 6.11. *The calculus $L(\backslash)$ has the generalized interpolation property.*

7. Construction of a context-free grammar for a product-free Lambek grammar

In this section we consider categorial grammars based on the product-free fragment of the Lambek calculus.

DEFINITION 7.1. A *product-free Lambek grammar* is a triple $\langle \mathcal{T}, H, \triangleright\rangle$, where $\mathcal{T}$ is a finite set (the alphabet), $H \in \mathrm{Tp}(\backslash, /)$, and $\triangleright$ is a finite binary relation $\triangleright \subset \mathrm{Tp}(\backslash, /) \times \mathcal{T}$.

The *language generated by the product-free Lambek grammar* $\langle \mathcal{T}, H, \triangleright\rangle$ is defined as the set of all strings $t_1 \dots t_n$ over the alphabet $\mathcal{T}$ for which there exist types $B_1, \dots, B_n \in \mathrm{Tp}(\backslash, /)$ such that $L(\backslash, /) \vdash B_1 \dots B_n \to H$ and $B_i \triangleright t_i$ for all $i \leq n$.

The cut-elimination property of the Lambek calculus entails its conservativity over its elementary fragments. Thus Theorem 5.11 implies that all languages generated by product-free Lambek grammars are context-free. However, in general the construction used in the proof of Theorem 5.11 involves types with product (as non-terminal symbols).

The following question arises. Is it possible to construct for arbitrary product-free Lambek grammar a corresponding "natural" context-free grammar with a finite subset of $\mathrm{Tp}(\backslash, /)$ as the alphabet of non-terminal symbols? The positive answer to this question is given by Theorem 7.3.

LEMMA 7.2. *Let a thin sequent $\Phi\Theta\Psi \to C$ be derivable in $L(\backslash, /)$ and the sequence $E_1 \dots E_r \in \mathrm{Tp}(\backslash, /)$ be an interpolant corresponding to a partition $\Theta = \Theta_1 \dots \Theta_r$ of the sequence Θ in the sequent $\Phi[\Theta]\Psi \to C$. Then*

(i) *for every $i \leq r$ the sequent $\Theta_i \to E_i$ is thin;*
(ii) *the sequent $\Phi E_1 \dots E_r \Psi \to C$ is thin;*
(iii) $\|E_1 \dots E_r\| = |[\![\Theta]\!]|$.

PROOF. Similar to Lemma 4.5. □

THEOREM 7.3. *Let $\langle \mathcal{T}, H, \triangleright\rangle$ be a product-free Lambek grammar. We put*

$$\begin{aligned}
\mathcal{U} &\rightleftharpoons \{H\} \cup \{\|B\| \mid \textit{there is } t \in \mathcal{T} \textit{ such that } B \triangleright t\};\\
m &\rightleftharpoons \max\{\|A\| \mid A \in \mathcal{U}\};\\
s &\rightleftharpoons \max\{i \in \mathbb{N} \mid \textit{there is } A \in \mathcal{U} \textit{ such that } p_i \in \mathrm{Var}(A)\};\\
\mathcal{W} &\rightleftharpoons \{A \in \mathrm{Tp}(\backslash, /) \mid \mathrm{Var}(A) \subseteq \{p_1, \dots, p_s\} \textit{ and } \|A\| \leq m\};\\
\sigma &\rightleftharpoons H;\\
\mathcal{R} &\rightleftharpoons \{B \Rightarrow t \mid t \in \mathcal{T} \textit{ and } B \triangleright t\} \cup\\
&\qquad \{A \Rightarrow \Gamma \mid L \vdash \Gamma \to A,\ A \in \mathcal{W},\ \Gamma \in \mathcal{W}^+,\ \textit{and } \|\Gamma\| \leq 2m\}.
\end{aligned}$$

Then the context-free grammar $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R}\rangle$ and the given Lambek grammar $\langle \mathcal{T}, H, \triangleright\rangle$ generate the same language.

PROOF. One can repeat the argument from Section 5. The only non-trivial part is the proof of Lemma 5.9. We use Lemma 7.2 instead of Lemma 4.5 and obtain a type sequence $E_1 \dots E_r$ as an interpolant. After this the cut rule needs to be applied r times. □

8. Conjoinable types in the Lambek calculus

The notion of conjoinability for the Lambek calculus is defined in [**2**, p. 76] as follows.

Definition 8.1. Two types A and B are said to be *conjoinable* iff there exists some type C such that $L \vdash A \to C$ and $L \vdash B \to C$.

In this section it will be proved that two types A and B are conjoinable if and only if $[\![A]\!] = [\![B]\!]$.

Definition 8.2. We say that two types A and B are *interchangeable* if and only if $L \vdash A \to B$ and $L \vdash B \to A$.

Lemma 8.3. *The interchangeability relation is a congruence on types.*

Proof. Immediate from admissibility of the following rules.

$$\frac{A \to B}{A{\bullet}C \to B{\bullet}C} \qquad \frac{A \to B}{C{\bullet}A \to C{\bullet}B}$$

$$\frac{A \to B}{B\backslash C \to A\backslash C} \qquad \frac{A \to B}{C\backslash A \to C\backslash B}$$

$$\frac{A \to B}{A/C \to B/C} \qquad \frac{A \to B}{C/B \to C/A}$$

□

Lemma 8.4.

(i) *The types $(A\backslash B)/C$ and $A\backslash(B/C)$ are interchangeable.*
(ii) *The types $(A{\bullet}B){\bullet}C$ and $A{\bullet}(B{\bullet}C)$ are interchangeable.*

We shall write $A\backslash B/C$ and $A{\bullet}B{\bullet}C$ (omitting unnecessary parentheses).

Lemma 8.5. *Let A and B be any two types of the Lambek calculus. The following three assertions are equivalent.*

(i) *There exists a type C such that $L \vdash A \to C$ and $L \vdash B \to C$.*
(ii) *There exists a type D such that $L \vdash D \to A$ and $L \vdash D \to B$.*
(iii) *There exist types $C_0, \ldots, C_n$ such that $A = C_0$, $B = C_n$, and for every $i < n$, $L \vdash C_i \to C_{i+1}$ or $L \vdash C_{i+1} \to C_i$.*

Proof. This Lemma is proved by J. Lambek in [**10**]. To derive (ii) form (i), put $D \rightleftharpoons (A/C){\bullet}C{\bullet}(C\backslash B)$. For the converse one can use $C \rightleftharpoons (D/A)\backslash D/(B\backslash D)$. □

Lemma 8.6. *The conjoinability relation is a congruence on types.*

Proof. Similar to the proof of Lemma 8.3. □

Lemma 8.7.

(i) *The types $A\backslash A$ and B/B are conjoinable.*
(ii) *The types $A\backslash A$ and $B\backslash B$ are conjoinable.*
(iii) *The types B and $B/(A\backslash A)$ are conjoinable.*
(iv) *The types B and $B{\bullet}(A\backslash A)$ are conjoinable.*
(v) *The types B/A and $B{\bullet}(A\backslash A/A)$ are conjoinable.*
(vi) *The types A and $(A\backslash A/A)\backslash(A\backslash A/A)/(A\backslash A/A)$ are conjoinable.*

PROOF. (i) Note that $L \vdash A\backslash A \to A\backslash(A\bullet B)/B$ and $L \vdash B/B \to A\backslash(A\bullet B)/B$.

(ii) Follows from (i) by transitivity.

(iii) According to (ii), $B/(B\backslash B)$ and $B/(A\backslash A)$ are conjoinable. On the other hand, $L \vdash B \to B/(B\backslash B)$.

(iv) According to (ii), $B\bullet(B\backslash B)$ and $B\bullet(A\backslash A)$ are conjoinable. On the other hand, $L \vdash B\bullet(B\backslash B) \to B$.

(v) From (iv) we conclude that B/A and $(B\bullet(A\backslash A))/A$ are conjoinable. On the other hand, $L \vdash B\bullet(A\backslash A/A) \to (B\bullet(A\backslash A))/A$.

(vi) $L \vdash A \to (A\backslash A/A)\backslash(A\backslash A/A)/(A\backslash A/A)$. □

We introduce the auxiliary notion of *simple product* and construct a function from $F(\mathrm{Var})$ into the set of simple products.

DEFINITION 8.8. A *simple product* is any type which is a product of factors of the form p and $(p\backslash p)/p$, where $p \in \mathrm{Var}$. The set of all simple products will be denoted by SP.

DEFINITION 8.9. We define the function $\mathrm{sp}\colon F(\mathrm{Var}) \to \mathrm{SP}$ as follows:

$$\begin{aligned} \mathrm{sp}(\varepsilon) &\rightleftharpoons ((p_1\backslash p_1)/p_1)\bullet p_1, \\ \mathrm{sp}(p) &\rightleftharpoons p, \\ \mathrm{sp}(p^{-1}) &\rightleftharpoons (p\backslash p)/p, \\ \mathrm{sp}(uv) &\rightleftharpoons \mathrm{sp}(u)\bullet\mathrm{sp}(v) \text{ if } |u| = 1 \text{ and } |v| \geq 1. \end{aligned}$$

LEMMA 8.10. *The following claims hold for any $u \in F(\mathrm{Var})$ and for arbitrary types A and B.*

(i) *The types $B/\mathrm{sp}(u)$ and $B\bullet\mathrm{sp}(u^{-1})$ are conjoinable.*
(ii) *The types $\mathrm{sp}(u)\backslash B$ and $\mathrm{sp}(u^{-1})\bullet B$ are conjoinable.*
(iii) *The types A and $\mathrm{sp}(\llbracket A\rrbracket)$ are conjoinable.*

PROOF. First we prove (i) by induction on the length of u. Induction base consists of three cases.

Case $u = \varepsilon$. Note that $\mathrm{sp}(\varepsilon)$ and $p_1\backslash p_1$ are conjoinable. It remains to prove that $B/(p_1\backslash p_1)$ and $B\bullet(p_1\backslash p_1)$ are conjoinable. This is immediate from Lemma 8.7 (iii) and (iv).

Case $u = p$ follows from Lemma 8.7 (v).

Case $u = p^{-1}$. From Lemma 8.7 (v) we obtain that $B/(p\backslash p/p)$ and $B\bullet((p\backslash p/p)\backslash(p\backslash p/p)/(p\backslash p/p))$ are conjoinable. In view of Lemma 8.7 (vi), $B/(p\backslash p/p)$ and $B\bullet p$ are conjoinable.

Induction step. Let $u = vw$, where $|w| \geq 1$ and v is either p or p^{-1}. We must prove that $B/(\mathrm{sp}(v)\bullet\mathrm{sp}(w))$ and $B\bullet\mathrm{sp}(w^{-1})\bullet\mathrm{sp}(v^{-1})$ are conjoinable. Note that $B/(\mathrm{sp}(v)\bullet\mathrm{sp}(w))$ and $(B/\mathrm{sp}(w))/\mathrm{sp}(v)$ are conjoinable (they are even interchangeable). According to induction hypothesis $B/\mathrm{sp}(w)$ and $B\bullet\mathrm{sp}(w^{-1})$ are conjoinable. It remains to establish that $(B\bullet\mathrm{sp}(w^{-1}))/\mathrm{sp}(v)$ and $B\bullet\mathrm{sp}(w^{-1})\bullet\mathrm{sp}(v^{-1})$ are conjoinable. But this follows from induction base for $B' = B\bullet\mathrm{sp}(w^{-1})$.

One can prove (ii) dually. The clause (iii) is proved by induction on the length of A, using (i) and (ii). □

THEOREM 8.11. *Two types A and B are conjoinable if and only if $\llbracket A\rrbracket = \llbracket B\rrbracket$.*

PROOF. If $L \vdash A \to C$ and $L \vdash B \to C$, then $\llbracket A\rrbracket = \llbracket B\rrbracket$ in view of Lemma 2.3. The converse implication follows from Lemma 8.10 (iii). □

9. Multiplicative cyclic linear logic

All results in this section are proved similarly to corresponding Lambek calculus results from preceding sections.

9.1. The calculus CLL. The cyclic linear logic was introduced in [**21**]. Here we consider its multiplicative fragment and denote it by CLL.

A countable set $\mathrm{Var} = \{p_1, p_2, p_3, \ldots\}$ is assumed to be given. In linear logic setting we shall call the elements of this set *atomic formulas*. They play precisely the same role as primitive types in the Lambek calculus.

The set of formulas $\mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)$ of the calculus CLL is defined as the smallest set satisfying the following conditions:

- $\mathbf{1} \in \mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)$ and $\bot \in \mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)$;
- if $p_i \in \mathrm{Var}$, then $p_i \in \mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)$ and $p_i^\bot \in \mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)$;
- if $A \in \mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)$ and $B \in \mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)$, then $(A \bullet B) \in \mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)$ and $(A \parr B) \in \mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)$.

The sequents of the calculus CLL are of the form $\to \Gamma$, where $\Gamma \in \mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)^*$.

We need an operation $(\ \cdot\ )^\bot \colon \mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot) \to \mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)$ defined on the set $\mathrm{Fm}(\bullet, \parr, \mathbf{1}, \bot)$. It maps each formula to its *negation*.

$$
\begin{aligned}
(\mathbf{1})^\bot &\rightleftharpoons \bot\\
(\bot)^\bot &\rightleftharpoons \mathbf{1}\\
(p_i)^\bot &\rightleftharpoons p_i^\bot\\
(p_i^\bot)^\bot &\rightleftharpoons p_i\\
(A \bullet B)^\bot &\rightleftharpoons ((B)^\bot \parr (A)^\bot)\\
(A \parr B)^\bot &\rightleftharpoons ((B)^\bot \bullet (A)^\bot)
\end{aligned}
$$

We shall write $CLL \vdash \Gamma$ iff the sequent $\to \Gamma$ is derivable in CLL. In this section $A_1 \ldots A_n \to B$ will stand for $\to (A_n)^\bot \ldots (A_1)^\bot\, B$.

The axioms of the calculus CLL are all sequents of the form $\to (p_i)^\bot\, p_i$, where $p_i \in \mathrm{Var}$, as well as the sequent $\to \mathbf{1}$.

The calculus CLL has the following derivation rules.

$$
\frac{\to \Gamma A B \Delta}{\to \Gamma (A \parr B) \Delta}\ (\to\parr)
\qquad
\frac{\to \Gamma A \quad \to B \Delta}{\to \Gamma (A \bullet B) \Delta}\ (\to \bullet)
$$

$$
\frac{\to \Gamma \Delta}{\to \Gamma \bot \Delta}\ (\to \bot)
\qquad
\frac{\to \Gamma \Delta}{\to \Delta \Gamma}\ (\text{rotate})
$$

$$
\frac{\to \Gamma A \quad \to (A)^\bot\, \Delta}{\to \Gamma \Delta}\ (\text{cut})
$$

Remark 9.1. The calculus CLL is conservative over the calculus L^* if we translate $A \backslash B$ as $(A)^\bot \parr B$ and B / A as $B \parr (A)^\bot$. If we constrain the derivation rule $(\to\parr)$ requiring that $\Gamma\Delta \neq \Lambda$ and omit the axiom $\to \mathbf{1}$, then we obtain a variant of the cyclic linear logic, which is conservative over the Lambek calculus L.

9.2. Free group interpretation of CLL-formulas.

Definition 9.2. The *free group interpretation of CLL-formulas* (denoted by $[\![\ ]\!]$) is the following natural mapping of formulas and their finite sequences into the group $F(\mathrm{Var})$:

$$\begin{array}{rcl}
[\![\mathbf{1}]\!] & \rightleftharpoons & \varepsilon \\
[\![\bot]\!] & \rightleftharpoons & \varepsilon \\
[\![p_i]\!] & \rightleftharpoons & p_i \\
[\![p_i^{\bot}]\!] & \rightleftharpoons & p_i^{-1} \\
[\![A \bullet B]\!] & \rightleftharpoons & [\![A]\!][\![B]\!] \\
[\![A \invamp B]\!] & \rightleftharpoons & [\![A]\!][\![B]\!]
\end{array}$$

$$[\![A_1 \dots A_n]\!] \rightleftharpoons [\![A_1]\!] \dots [\![A_n]\!].$$

LEMMA 9.3. *For any formula* $A \in \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)$, $|[\![A]\!]| \leq \|A\|$.

PROOF. By induction on the construction of A. □

LEMMA 9.4. *If a sequent* $\to \Gamma$ *is derivable in the calculus* CLL*, then* $[\![\Gamma]\!] = \varepsilon$.

PROOF. By induction on derivations similarly to the proof of Lemma 2.3. □

9.3. Thin sequents in CLL.

DEFINITION 9.5. The *length* $\|A\|$ of a formula A is defined as the total number of atomic formula occurrences in A.

$$\begin{array}{rcl}
\|\mathbf{1}\| & \rightleftharpoons & 0 \\
\|\bot\| & \rightleftharpoons & 0 \\
\|p_i\| & \rightleftharpoons & 1 \\
\|p_i^{\bot}\| & \rightleftharpoons & 1 \\
\|A \bullet B\| & \rightleftharpoons & \|A\| + \|B\| \\
\|A \invamp B\| & \rightleftharpoons & \|A\| + \|B\|
\end{array}$$

The length of a finite sequence of formulas is defined in the natural way:

$$\|A_1 \dots A_n\| \rightleftharpoons \|A_1\| + \dots + \|A_n\|.$$

DEFINITION 9.6. The set of all atomic formulas occurring in a formula A will be denoted by $\mathrm{Var}(A)$.

DEFINITION 9.7. For every atomic formula $p \in \mathrm{Var}$ we define two mappings $\#_p^+$ and $\#_p^-$ from the set $\mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)$ into $\mathbb{N}$.

$$\begin{array}{rcl}
\#_p^+(q) & \rightleftharpoons & \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } q \in \mathrm{Var} \text{ and } p \neq q \end{cases} \\
\#_p^-(q) & \rightleftharpoons & 0, \text{if } q \in \mathrm{Var} \\
\#_p^+(q^{\bot}) & \rightleftharpoons & 0, \text{if } q \in \mathrm{Var} \\
\#_p^-(q^{\bot}) & \rightleftharpoons & \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } q \in \mathrm{Var} \text{ and } p \neq q \end{cases} \\
\#_p^+(A \bullet B) & \rightleftharpoons & \#_p^+(A) + \#_p^+(B) \\
\#_p^-(A \bullet B) & \rightleftharpoons & \#_p^-(A) + \#_p^-(B) \\
\#_p^+(A \invamp B) & \rightleftharpoons & \#_p^+(A) + \#_p^+(B) \\
\#_p^-(A \invamp B) & \rightleftharpoons & \#_p^-(A) + \#_p^-(B)
\end{array}$$

These mappings are extended to finite sequences of formulas.

$$\#_p^+(A_1 \dots A_n) \;\rightleftharpoons\; \#_p^+(A_1) + \dots + \#_p^+(A_n)$$
$$\#_p^-(A_1 \dots A_n) \;\rightleftharpoons\; \#_p^-(A_1) + \dots + \#_p^-(A_n)$$

Definition 9.8. A sequent $\to \Pi$ is *thin* iff $\#_p^+(\Pi) \le 1$ and $\#_p^-(\Pi) \le 1$ for every $p \in \mathrm{Var}$.

Lemma 9.9. *Let ϕ be an atomic formula substitution. If one replaces every sequent $\to \Gamma$ by $\to \phi(\Gamma)$ in a CLL-derivation, then the resulting tree is a legal CLL-derivation.*

Theorem 9.10. *A sequent $\to \Pi$ is derivable in CLL if and only if there exists a derivable in CLL thin sequent $\to \Theta$ and there exists an atomic formula substitution ϕ such that $\Pi = \phi(\Theta)$.*

9.4. Interpolation in CLL.

Lemma 9.11. *Let $CLL \vdash \Gamma\Pi\Delta$, $\Gamma \in \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)^*$, $\Pi \in \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)^*$, and $\Delta \in \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)^*$. Then there exists a formula E such that*
- (i) $CLL \vdash (E)^\bot\, \Pi$;
- (ii) $CLL \vdash \Gamma E \Delta$;
- (iii) *the inequality $\#_p^+(E) \le \min(\#_p^+(\Pi), \#_p^-(\Gamma\Delta))$ holds for every atomic formula p;*
- (iv) *the inequality $\#_p^-(E) \le \min(\#_p^-(\Pi), \#_p^+(\Gamma\Delta))$ holds for every atomic formula p.*

Lemma 9.12. *Let $CLL \vdash \Gamma\Pi\Delta$, $\Gamma \in \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)^*$, $\Pi \in \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)^*$, $\Delta \in \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)^*$. Let the sequent $\to \Gamma\Pi\Delta$ be thin. Then there exists a formula E such that*
- (i) $CLL \vdash (E)^\bot\, \Pi$;
- (ii) $CLL \vdash \Gamma E \Delta$;
- (iii) *the sequent $\to (E)^\bot\, \Pi$ is thin;*
- (iv) *the sequent $\to \Gamma E \Delta$ is thin;*
- (v) $\|E\| = |[\![\Pi]\!]|$.

Lemma 9.13. *Let $CLL \vdash \Phi\Theta\Psi \to C$, $\Phi \in \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)^*$, $\Theta \in \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)^*$, $\Psi \in \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)^*$, $C \in \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot)$. Let the sequent $\Phi\Theta\Psi \to C$ be thin. Then there exists a formula E such that*
- (i) $CLL \vdash \Theta \to E$;
- (ii) $CLL \vdash \Phi E \Psi \to C$;
- (iii) *the sequent $\Theta \to E$ is thin;*
- (iv) *the sequent $\Phi E \Psi \to C$ is thin;*
- (v) $\|E\| = |[\![\Theta]\!]|$.

Proof. Given $CLL \vdash (\Psi)^\bot\, (\Theta)^\bot\, (\Phi)^\bot\, C$, it remains to apply Lemma 9.12 with $\Gamma \rightleftharpoons (\Psi)^\bot$, $\Pi \rightleftharpoons (\Theta)^\bot$, $\Delta \rightleftharpoons (\Phi)^\bot\, C$. □

9.5. Grammars based on CLL.

Definition 9.14. A *categorial grammar based on the calculus CLL* (or a CLL-grammar) is a triple $\langle \mathcal{T}, H, \triangleright \rangle$, where $\mathcal{T}$ is a finite set (the alphabet), H is a formula, and $\triangleright$ is a finite binary relation $\triangleright \subset \mathrm{Fm}(\bullet, \invamp, \mathbf{1}, \bot) \times \mathcal{T}$.

The *language generated by the grammar* $\langle \mathcal{T}, H, \triangleright \rangle$ is defined as the set of all non-empty strings $t_1 \dots t_n$ over the alphabet $\mathcal{T}$ for which there exists a derivable in CLL sequent $B_1 \dots B_n \to H$ such that $B_i \triangleright t_i$ for all $i \le n$. We shall denote this language by $\mathcal{L}_{CLL}(\mathcal{T}, H, \triangleright)$.

Remark 9.15. It is possible that the sequent $\to H$ is derivable in CLL. Nevertheless the empty word is not included in the language generated by the grammar. This ensures compatibility with our definition of a context-free grammar at p. 4, where we banned productions of the form $\alpha \Rightarrow \varepsilon$ and thus excluded the possibility that the empty word would occur in the generated language.

9.6. Context-freeness of CLL-grammars.

Definition 9.16. We introduce two families of sets of CLL-formulas.

$$\begin{array}{rcl} \mathrm{Fm}(m) & \rightleftharpoons & \{A \in \mathrm{Fm}(\bullet, \wp, \mathbf{1}, \bot) \mid \|A\| \le m\} \\ \mathrm{Fm}(m,s) & \rightleftharpoons & \{A \in \mathrm{Fm}(m) \mid \mathrm{Var}(A) \subseteq \{p_1, \dots, p_s\}\} \end{array}$$

Consider an arbitrary CLL-grammar $\langle \mathcal{T}, H, \triangleright \rangle$. Only a finite number of types are relevant in the definition of the language generated by this grammar. Thus there are positive integers m and s such that $H \in \mathrm{Fm}(m,s)$ and if $B \triangleright t$ for some $t \in \mathcal{T}$, then $B \in \mathrm{Fm}(m,s)$.

There is no loss of generality in assuming that the sets $\mathcal{T}$ and $\mathrm{Fm}(m,s)$ do not intersect. Now we construct the desired context-free grammar $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R} \rangle$.

$\mathcal{W} \rightleftharpoons \mathrm{Fm}(m,s)$

$\sigma \rightleftharpoons H$

$$\begin{array}{l} \mathcal{R} \rightleftharpoons \{B \Rightarrow t \mid t \in \mathcal{T} \text{ and } B \triangleright t\} \cup \\ \qquad \{A \Rightarrow \Gamma \mid A \in \mathrm{Fm}(m,s), \Gamma \in \mathrm{Fm}(m,s)^*, \|\Gamma\| \le 2m, \text{ and } CLL \vdash \Gamma \to A\} \end{array}$$

We define an auxiliary calculus $CLLcut_m$.

Definition 9.17. A sequent $\Gamma \to A$ is an axiom of the calculus $CLLcut_m$ if and only if $CLL \vdash \Gamma \to A$, $A \in \mathrm{Fm}(m)$, $\Gamma \in \mathrm{Fm}(m)^*$, and $\|\Gamma\| \le 2m$. The only derivation rule of the calculus $CLLcut_m$ is the cut rule

$$\frac{\Pi \to B \quad \Gamma B \Delta \to A}{\Gamma \Pi \Delta \to A.}\ (\text{cut})$$

Lemma 9.18. *Let $\langle \mathcal{T}, H, \triangleright \rangle$ be a CLL-grammar and $\langle \mathcal{T}, \mathcal{W}, \sigma, \mathcal{R} \rangle$ be the corresponding context-free grammar. Let $\Gamma \in \mathrm{Fm}(m,s)^*$ and $A \in \mathrm{Fm}(m,s)$. Then the following three assertions are equivalent:*

(i) $C_2(\mathcal{W}, \mathcal{R}) \vdash \Gamma \to A$;
(ii) $CLLcut_m \vdash \Gamma \to A$;
(iii) $CLL \vdash \Gamma \to A$.

Theorem 9.19. *Let $\langle \mathcal{T}, H, \triangleright \rangle$ be an arbitrary CLL-grammar. Then the language $\mathcal{L}_{CLL}(\mathcal{T}, H, \triangleright)$ is context-free.*

Remark 9.20. The converse is true in view of conservativity of CLL over L^*. Consequently the class of languages generated by grammars based on the multiplicative cyclic linear logic coincides with the class of all context-free languages.

Index

References

[1] Y. Bar-Hillel, C. Gaifman, and E. Shamir, *On categorial and phrase structure grammars*, Bull. Res. Council Israel F **9** (1960), 1–16.

[2] J. van Benthem, *Language in action*, North-Holland, Amsterdam, 1991.

[3] W. Buszkowski, *The equivalence of unidirectional Lambek categorial grammars and context-free grammars*, Z. Math. Logik Grundlagen Math. **31** (1985), 369–384.

[4] ———, *Generative power of categorial grammars*, Categorial Grammars and Natural Language Structures (R. T. Oehrle, E. Bach, and D. Wheeler, eds.), Reidel, Dordrecht, 1988, pp. 69–94.

[5] ———, *On generative capacity of the Lambek calculus*, Logics in AI (J. van Eijck, ed.), Springer, 1991, pp. 139–152.

[6] ———, *On the equivalence of Lambek categorial grammars and basic categorial grammars*, ILLC Prepublication Series LP–93–07, Institute for Logic, Language and Computation, University of Amsterdam, 1993.

[7] N. Chomsky, *Formal properties of grammars*, Handbook of Mathematical Psychology (R. D. Luce et al., eds.), vol. 2, John Wiley and Sons, New York, 1963, pp. 323–418.

[8] J. M. Cohen, *The equivalence of two concepts of categorial grammar*, Inform. and Control **10** (1967), 475–484.

[9] G. Lallement, *Semigroups and combinatorial applications*, John Wiley and Sons, New York, 1979.

[10] J. Lambek, *The mathematics of sentence structure*, Amer. Math. Monthly **65** (1958), no. 3, 154–170.

[11] M. Pentus, *Equivalent types in Lambek calculus and linear logic*, Prepublication Series in Logic and Computer Science LCS–92–2, Steklov Mathematical Institute, Moscow, April 1992.

[12] ———, *Lambek grammars are context free*, Prepublication Series in Logic and Computer Science LCS–92–8, Steklov Mathematical Institute, Moscow, December 1992.

[13] ———, *The conjoinability relation in Lambek calculus and linear logic*, ILLC Prepublication Series ML–93–03, Institute for Logic, Language and Computation, University of Amsterdam, 1993.

[14] ———, *Lambek grammars are context free*, Proceedings of the 8th Annual IEEE Symposium on Logic in Computer Science, IEEE Computer Society Press, 1993, pp. 429–433.

[15] ———, *The conjoinability relation in Lambek calculus and linear logic*, J. Logic Lang. Inform. **3** (1994), no. 2, 121–140.

[16] ———, *Lambek calculus and formal grammars*, Fundamentalnaya i prikladnaya matematika **1** (1995), no. 3, 729–751 (Russian).

[17] ———, *Product-free Lambek calculus and context-free grammars*, J. Symbolic Logic **62** (1997), no. 2, 648–660.

[18] D. Roorda, *Resource logics: proof-theoretical investigations*, Ph.D. thesis, University of Amsterdam, 1991.

[19] ———, *Interpolation in fragments of classical linear logic*, J. Symbolic Logic **59** (1994), no. 2, 419–444.

[20] K. Schütte, *Der Interpolationssatz der intuitionistischen Prädikatenlogik*, Math. Ann. **148** (1962), 192–200.

[21] D. N. Yetter, *Quantales and noncommutative linear logic*, J. Symbolic Logic **55** (1990), no. 1, 41–64.

Department of Mathematical Logic and Theory of Algorithms, Faculty of Mechanics and Mathematics, Moscow State University

E-mail address: `pentus@lpcs.math.msu.ru`

URL: `http://markov.math.msu.ru/~pentus/`