Models for the Lambek calculus

Mati Pentus*

Abstract

We prove that the Lambek calculus is complete w.r.t. L-models, i.e., free semigroup models. We also prove the completeness w.r.t. relativized relational models over the natural linear order of integers.

Introduction

In 1958 Lambek [8] introduced a calculus for deriving reduction laws of syntactic types. The inteneded syntactic string models, i.e., *free semigroup models* (also called *language models* or *L-models*) for this calculus were considered in [2], [3], and [4]. The more general class of *groupoid models* has been studied in [5], [6], and [7]. In [3] W. Buszkowski established that the product-free fragment of the Lambek calculus is L-complete (i.e., complete w.r.t. free semigroup models), using the canonical model. The question of L-completeness of the full Lambek calculus remained open (cf. [1]). At the end of 1992 the author of this paper gave a positive answer to this question. The proof was improved after several talks in Moscow, Amsterdam, Berne, Paris, issued as the preprint [12]; a brief exposition of it one may also find in [13]. In the current paper we deliver the complete proof essentially similar to that from [12], with some minor misprints corrected.

Another interesting particular case of groupoid semantics considered here is relational semantics. Mikulás proved in 1992 [9] that the Lambek calculus is complete w.r.t. relativized relational models (R-models) (cf. also [10]). Pankratiev [11] proved the completeness w.r.t. R-models over the left-divisor relation in a special residuated semigroup.

In this paper we prove the completeness of the Lambek calculus w.r.t. R-models on a very simple frame, namely on the natural order of integers.

1 Preliminaries

1.1 Lambek calculus

We consider the syntactic calculus introduced in [8]. The types of the Lambek calculus are built of primitive types $p_1, p_2, \ldots$, and three binary connectives $\bullet, \backslash, /$. We shall denote

*This research was supported in part by the International Science Foundation under Grant No. NFQ000.

the set of all types by Tp. The set of finite sequences of types (resp. finite non-empty sequences of types) is denoted by Tp^* (resp. Tp^+). The symbol Λ will stand for the empty sequence of types.

Capital letters A,B,... range over types. Capital Greek letters range over finite (possibly empty) sequences of types.

Sequents of the Lambek calculus are of the form $\Gamma\to A$, where Γ is a non-empty sequence of types.

Axioms: $A\to A$

Rules:

$$\frac{A\Pi\to B}{\Pi\to A\backslash B}\ (\to\backslash) \quad \text{where} \quad \Pi\neq\Lambda \qquad\qquad \frac{\Phi\to A \quad \Gamma B\Delta\to C}{\Gamma\Phi(A\backslash B)\Delta\to C}\ (\backslash\to)$$

$$\frac{\Pi A\to B}{\Pi\to B/A}\ (\to/) \quad \text{where} \quad \Pi\neq\Lambda \qquad\qquad \frac{\Phi\to A \quad \Gamma B\Delta\to C}{\Gamma(B/A)\Phi\Delta\to C}\ (/\to)$$

$$\frac{\Gamma\to A \quad \Delta\to B}{\Gamma\Delta\to A{\bullet}B}\ (\to\bullet) \qquad\qquad \frac{\Gamma AB\Delta\to C}{\Gamma(A{\bullet}B)\Delta\to C}\ (\bullet\to)$$

$$\frac{\Phi\to B \quad \Gamma B\Delta\to A}{\Gamma\Phi\Delta\to A}\ (CUT)$$

The cut-elimination theorem for this calculus is proved in [8]. We write $L\vdash\Gamma\to A$ if the sequent $\Gamma\to A$ is derivable in the Lambek calculus.

Definition. The *length* of a type is defined as the total number of primitive type occurreces in the type.

$$\|p_i\|\rightleftharpoons 1 \qquad \|A{\bullet}B\|\rightleftharpoons\|A\|+\|B\|$$

$$\|A\backslash B\|\rightleftharpoons\|A\|+\|B\| \qquad \|A/B\|\rightleftharpoons\|A\|+\|B\|$$

Similarly, for sequences of types we put $\|A_1\dots A_n\|\rightleftharpoons\|A_1\|+\dots+\|A_n\|$.

Definition. The set of primitive types *occurring* in a type is defined as follows.

$$\mathrm{Var}(p_i)\rightleftharpoons\{p_i\} \qquad \mathrm{Var}(A{\bullet}B)\rightleftharpoons\mathrm{Var}(A)\cup\mathrm{Var}(B)$$

$$\mathrm{Var}(A\backslash B)\rightleftharpoons\mathrm{Var}(A)\cup\mathrm{Var}(B) \qquad \mathrm{Var}(A/B)\rightleftharpoons\mathrm{Var}(A)\cup\mathrm{Var}(B)$$

Definition. For any integer m, we write $\mathrm{Tp}(m)$ for the finite set of types

$$\mathrm{Tp}(m)\rightleftharpoons\{A\in\mathrm{Tp}\mid\mathrm{Var}(A)\subseteq\{p_1,p_2,\dots,p_m\}\ \text{ and }\ \|A\|\leq m\}.$$

By $\mathrm{Tp}(m)^+$ we denote the set of all non-empty finite sequences of types from $\mathrm{Tp}(m)$.

Definition. For any two integers m and n, we write $\mathrm{LST}_{m,n}$ (limited sequences of types) for the following finite subset of $\mathrm{Tp}(m)^+$.

$$\mathrm{LST}_{m,n}\rightleftharpoons\{A_1\dots A_l\mid 1\leq l\leq n,\ A_1\in\mathrm{Tp}(m),\ \dots,\ A_l\in\mathrm{Tp}(m)\}$$

Definition. Sometimes we shall write ${\bullet}(A_1\dots A_n)$ or $A_1{\bullet}\dots{\bullet}A_n$ instead of $(\dots(A_1{\bullet}A_2){\bullet}\dots{\bullet}A_n)$.

1.2 Partial semigroup models

Definition. We say that $\langle \mathbf{W}, \circ \rangle$ is a partial semigroup iff $\circ$ is a partial function from $\mathbf{W} \times \mathbf{W}$ into $\mathbf{W}$ such that, whenever $a \circ (b \circ c)$ or $(a \circ b) \circ c$ is defined, the other expression is also defined and $a \circ (b \circ c) = (a \circ b) \circ c$. (We do not exclude the case $\mathbf{W} = \emptyset$.)

Example 1 Here are some examples of partial semigroups.

(a) Any semigroup.

(b) The free semigroup with a countable set of generators.
Here $\mathbf{W}$ is the set of all non-empty words over the alphabet $\{a_j \mid j \in \mathbf{N}\}$ ($\mathbf{N}$ stands for the set of all natural numbers). The binary operator $\circ$ is the concatenation of words.

(c) The free semigroup with two generators.

(d) A binary relational frame (*R-frame* for short).
Here $\mathbf{W} \subset \mathbf{D} \times \mathbf{D}$ and $\mathbf{W}$ is an irreflexive transitive binary relation over a domain $\mathbf{D}$.

$$\langle s_1, t_1 \rangle \circ \langle s_2, t_2 \rangle \rightleftharpoons \begin{cases} \langle s_1, t_2 \rangle & \text{if } t_1 = s_2 \\ \text{undefined} & \text{if } t_1 \neq s_2 \end{cases}$$

Here $s_1, t_1, s_2, t_2 \in \mathbf{D}$.

(e) The natural order on integers as a binary relational frame (we shall denote this R-frame by $\langle <_{\mathbf{Z}}, \circ \rangle$).

$$\mathbf{D} = \mathbf{Z}, \qquad \mathbf{W} = \{\langle s, t \rangle \mid s \in \mathbf{Z},\ t \in \mathbf{Z}, \text{ and } s < t\}$$

By $\mathbf{Z}$ we denote the set of all integers. The operation $\circ$ is defined as in (d).

(f) The natural order on a finite interval of integers $[p, q]$, where $p \in \mathbf{Z}$ and $q \in \mathbf{Z}$.

$$\mathbf{D}_{[p,q]} = [p, q] = \{s \in \mathbf{Z} \mid p \leq s \leq q\}$$

$$\mathbf{W}_{[p,q]} = \{\langle s, t \rangle \mid s \in \mathbf{Z},\ t \in \mathbf{Z}, \text{ and } p \leq s < t \leq q\}$$

The operation $\circ$ is defined as in (d).

We shall denote by $\mathcal{S}_{\mathbf{Z}}$ the class[1] of all partial semigroups from Example 1 (f).

Definition. Let $\langle \mathbf{W}, \circ \rangle$ be a partial semigroup.
We say that $\langle \mathbf{V}, * \rangle$ is a *sub-partial-semigroup of* $\langle \mathbf{W}, \circ \rangle$ iff

(1) $V \subseteq W$;

[1] In this paper 'class' and 'set' are synonyms.

(2) $*$ is the restriction of $\circ$ to V.

Remark. The associativiy law holds automatically in every sub-partial-semigroup.

Example 2 Every partial semigroup from $\mathcal{S}_{\mathbf{Z}}$ is a sub-partial-semigroup of $\langle <_{\mathbf{Z}}, \circ \rangle$.

We shall use the following shorthand notation. For any sets $\mathcal{R} \subseteq \mathbf{W}$ and $\mathcal{T} \subseteq \mathbf{W}$ we write

$$\mathcal{R} \circ \mathcal{T} \rightleftharpoons \{\gamma \in \mathbf{W} \mid \text{ there are } \alpha \in \mathcal{R} \text{ and } \beta \in \mathcal{T} \text{ such that } \alpha \circ \beta = \gamma\};$$

$$\mathcal{R} \circ \beta \rightleftharpoons \mathcal{R} \circ \{\beta\}; \qquad \alpha \circ \mathcal{T} \rightleftharpoons \{\alpha\} \circ \mathcal{T}.$$

We shall denote the set of all subsets of a set $\mathbf{W}$ by $\mathbf{P}(\mathbf{W})$.

Definition. A partial semigroup model $\langle \mathbf{W}, \circ, w \rangle$ is a partial semigroup $\langle \mathbf{W}, \circ \rangle$ together with a valuation w associating with each type of the Lambek calculus a subset of $\mathbf{W}$ (i.e., $w\colon \mathrm{Tp} \to \mathbf{P}(\mathbf{W})$) and satisfying for any types A and B the following conditions.

(1) $w(A \bullet B) = w(A) \circ w(B)$

(2) $w(A \backslash B) = \{\gamma \in \mathbf{W} \mid \text{for all } \alpha \in w(A), \text{ if } \alpha \circ \gamma \text{ is defined then } \alpha \circ \gamma \in w(B)\}$

(3) $w(B / A) = \{\gamma \in \mathbf{W} \mid \text{for all } \alpha \in w(A), \text{ if } \gamma \circ \alpha \text{ is defined then } \gamma \circ \alpha \in w(B)\}$

Remark. One can reformulate (2) and (3) as follows.

(2') $w(A \backslash B) = \{\gamma \in \mathbf{W} \mid w(A) \circ \gamma \in w(B)\}$

(3') $w(B / A) = \{\gamma \in \mathbf{W} \mid \gamma \circ w(A) \in w(B)\}$

For any valuation w and for any types $A_1, \ldots, A_n$, we write $\overline{w}(A_1 \ldots A_n)$ as a shorthand for $w(A_1) \circ \ldots \circ w(A_n)$.

Definition. A sequent $\Gamma \to B$ is *true* in a model $\langle \mathbf{W}, \circ, w \rangle$ iff $\overline{w}(\Gamma) \subseteq w(B)$.
A sequent is *false* in a model iff it is not true in the model.

Definition. A partial semigroup model $\langle \mathbf{W}, \circ, w \rangle$ is called an *R-model* iff $\langle \mathbf{W}, \circ \rangle$ is an R-frame (cf. Example 1 (d)).

Remark. Partial semigroups form a subclass of associative ternary frames [7].

It is known that the Lambek calculus is sound w.r.t. associative ternary frames. Thus it is also sound w.r.t. all partial semigroup models, i.e., $\overline{w}(\Gamma) \subseteq w(B)$ for any partial semigroup model $\langle \mathbf{W}, \circ, w \rangle$, whenever $L \vdash \Gamma \to B$. On the other hand, W. Buszkowski [5] has proved that the Lambek calculus is complete w.r.t. models over arbitrary semigroups (Example 1 (a)). The completeness w.r.t. models over binary relational frames (Example 1 (d)) has been proved by Sz. Mikulás [9].

In this paper we prove that the Lambek calculus is also complete w.r.t. smaller classes of models, namely the models over the partial semigroups presented in Example 1 (b), (c) and (e).

The problem of completeness w.r.t. finite linear R-models (i.e., models over R-frames from $\mathcal{S}_{\mathbf{Z}}$) is still open.

2 Quasimodels

In this section we introduce the notion of $\mathrm{Tp}(m)$-quasimodels and describe an algorithm of constructing a partial semigroup model as the limit of an infinite sequence of $\mathrm{Tp}(m)$-quasimodels, which are conservative extensions of each other.

Definition. A *quasimodel* $\langle \mathbf{W}, \circ, w \rangle$ is a valuation w over a partial semigroup $\langle \mathbf{W}, \circ \rangle$ such that

(1) $w(A \bullet B) = w(A) \circ w(B)$ for any $A \in \mathrm{Tp}$, $B \in \mathrm{Tp}$;

(2) if $L \vdash A \to B$ then $w(A) \subseteq w(B)$.

Remark. Every partial semigroup model is a quasimodel.

Definition. A $\mathrm{Tp}(m)$-*quasimodel* $\langle \mathbf{W}, \circ, w \rangle$ is a valuation w over a partial semigroup $\langle \mathbf{W}, \circ \rangle$ such that

(1) if $A \bullet B \in \mathrm{Tp}(m)$, then $w(A \bullet B) = w(A) \circ w(B)$;

(2) if $\Gamma \in \mathrm{Tp}(m)^+$, $B \in \mathrm{Tp}(m)$, and $L \vdash \Gamma \to B$, then $\overline{w}(\Gamma) \subseteq w(B)$.

Remark. In the definition of a $\mathrm{Tp}(m)$-quasimodel the condition (1) can be replaced by (1').

(1') If $A \bullet B \in \mathrm{Tp}(m)$, then $w(A \bullet B) \subseteq w(A) \circ w(B)$.

(Note that $w(A) \circ w(B) \subseteq w(A \bullet B)$ follows from (2)).

Lemma 2.1 *Every quasimodel is a* $\mathrm{Tp}(m)$*-quasimodel for any* m.

PROOF. (1) is obvious. To prove (2) we assume $L \vdash A_1 \ldots A_l \to B$ and verify that $w(A_1) \circ \ldots \circ w(A_l) \subseteq w(B)$, where $A_1 \in \mathrm{Tp}(m)$, ..., $A_l \in \mathrm{Tp}(m)$, and $B \in \mathrm{Tp}(m)$.

Evidently $w(A_1) \circ \ldots \circ w(A_l) = w(A_1 \bullet \ldots \bullet A_l)$. Note that $L \vdash A_1 \bullet \ldots \bullet A_l \to B$, whence $w(A_1 \bullet \ldots \bullet A_l) \subseteq w(B)$. ■

Definition. A sequent $\Gamma \to A$ is *true* in a quasimodel (resp. $\mathrm{Tp}(m)$-quasimodel) $\langle \mathbf{W}, \circ, w \rangle$ iff $\overline{w}(\Gamma) \subseteq w(A)$.

Definition. A $\mathrm{Tp}(m)$-quasimodel $\langle \mathbf{W}, \circ, w \rangle$ is a *conservative extension* of another $\mathrm{Tp}(m)$-quasimodel $\langle \mathbf{V}, \circ, v \rangle$ iff

(1) $\langle \mathbf{V}, \circ \rangle$ is a sub-partial-semigroup of $\langle \mathbf{W}, \circ \rangle$;

(2) $w(A) \cap \mathbf{V} = v(A)$ for any type A.

Remark. The condition (2) can be reformulated in the following way.

(2') For any $\alpha \in \mathbf{V}$ and for any type A, $\alpha \in v(A)$ if and only if $\alpha \in w(A)$.

Remark. If $\langle \mathbf{W}, \circ, w \rangle$ is a conservative extension of $\langle \mathbf{V}, \circ, v \rangle$, then $v(A) \subseteq w(A)$ for every $A \in \mathrm{Tp}$.

Lemma 2.2 *If $\langle \mathbf{W}_2, \circ, w_2 \rangle$ is a conservative extension of $\langle \mathbf{W}_1, \circ, w_1 \rangle$ and $\langle \mathbf{W}_3, \circ, w_3 \rangle$ is a conservative extension of $\langle \mathbf{W}_2, \circ, w_2 \rangle$, then $\langle \mathbf{W}_3, \circ, w_3 \rangle$ is a conservative extension of $\langle \mathbf{W}_1, \circ, w_1 \rangle$.*

Proof. Evidently $\mathbf{W}_1$ is a sub-partial-semigroup of $\mathbf{W}_3$.

In view of $\mathbf{W}_1 \subseteq \mathbf{W}_2$ we have $w_3(A) \cap \mathbf{W}_1 = w_3(A) \cap (\mathbf{W}_2 \cap \mathbf{W}_1) = (w_3(A) \cap \mathbf{W}_2) \cap \mathbf{W}_1$. Further, $(w_3(A) \cap \mathbf{W}_2) \cap \mathbf{W}_1 = w_2(A) \cap \mathbf{W}_1 = w_1(A)$. ■

Definition. We say that a sequence of $\mathrm{Tp}(m)$-quasimodels $\langle \mathbf{W}_i, \circ, w_i \rangle$ $(i \in \mathbf{N})$ is *conservative* iff, for every $i \in \mathbf{N}$, $\langle \mathbf{W}_{i+1}, \circ, w_{i+1} \rangle$ is a conservative extension of $\langle \mathbf{W}_i, \circ, w_i \rangle$. (Here m is constant.)

Definition. The *limit* of a conservative sequence $\langle \mathbf{W}_i, \circ, w_i \rangle$ $(i \in \mathbf{N})$ is the $\mathrm{Tp}(m)$-quasimodel $\langle \mathbf{W}_\infty, \circ, w_\infty \rangle$ defined as follows.

(i) $\mathbf{W}_\infty \rightleftharpoons \bigcup_{i \in \mathbf{N}} \mathbf{W}_i$

(ii) $w_\infty(A) \rightleftharpoons \bigcup_{i \in \mathbf{N}} w_i(A)$

Lemma 2.3 *The definition of the limit is correct, i.e., $\langle \mathbf{W}_\infty, \circ, w_\infty \rangle$ is really a $\mathrm{Tp}(m)$-quasimodel.*

Proof.

(1) Proof of $\subseteq$.

Let $A \bullet B \in \mathrm{Tp}(m)$ and $\gamma \in w_\infty(A \bullet B)$. Then $\gamma \in w_n(A \bullet B) = w_n(A) \circ w_n(B)$ for some n. Thus $\gamma = \alpha \circ \beta$, where $\alpha \in w_n(A)$ and $\beta \in w_n(B)$. Evidently $\alpha \in w_\infty(A)$ and $\beta \in w_\infty(B)$, whence $\gamma = \alpha \circ \beta \in w_\infty(A) \circ w_\infty(B)$.

(1) Proof of $\supseteq$.

Let $A \bullet B \in \mathrm{Tp}(m)$ and $\gamma \in w_\infty(A) \circ w_\infty(B)$. Then $\gamma = \alpha \circ \beta$, where $\alpha \in w_\infty(A)$ and $\beta \in w_\infty(B)$, i.e., $\alpha \in w_i(A)$ and $\beta \in w_j(B)$ for some i and j. Put $n \rightleftharpoons \max(i, j)$.

Note that $\langle \mathbf{W}_n, \circ, w_n \rangle$ is a conservative extension of $\langle \mathbf{W}_i, \circ, w_i \rangle$. Hence $\alpha \in w_n(A)$. Similarly $\beta \in w_n(B)$. Thus $\gamma = \alpha \circ \beta \in w_n(A) \circ w_n(B) = w_n(A \bullet B) \subseteq w_\infty(A \bullet B)$.

(2)

Let $L \vdash A_1 \ldots A_l \to B$, where $A_1 \in \mathrm{Tp}(m)$, $\ldots$, $A_l \in \mathrm{Tp}(m)$, and $B \in \mathrm{Tp}(m)$. Assume that $\gamma \in \overline{w_\infty}(A_1 \ldots A_l)$, i.e., $\gamma = \alpha_1 \circ \ldots \circ \alpha_l$, where $\alpha_1 \in w_\infty(A_1)$, $\ldots$, $\alpha_l \in w_\infty(A_l)$. Then $\alpha_1 \in w_{i_1}(A_1)$, $\ldots$, $\alpha_l \in w_{i_l}(A_l)$ for some $i_1, \ldots, i_l \in \mathbf{N}$. Put $n \rightleftharpoons \max(i_1, \ldots, i_l)$.

Evidently $\alpha_1 \in w_n(A_1)$, $\ldots$, $\alpha_l \in w_n(A_l)$, whence $\gamma = \alpha_1 \circ \ldots \circ \alpha_l \in \overline{w_n}(A_1 \ldots A_l) \subseteq w_n(B) \subseteq w_\infty(B)$. ■

Lemma 2.4 *The limit of a conservative sequence is a conservative extension of any of the elements of the sequence.*

Proof. We verify that $w_\infty(A) \cap \mathbf{W}_i = w_i(A)$. If $i \leq j$ then $w_i(A) \subseteq w_j(A)$. Thus $w_\infty(A) = \bigcup_j w_j(A) = \bigcup_{j\geq i} w_j(A)$, whence $w_\infty(A)\cap\mathbf{W}_i = (\bigcup_{j\geq i} w_j(A))\cap\mathbf{W}_i = \bigcup_{j\geq i}(w_j(A)\cap\mathbf{W}_i)$. Note that $w_j(A) \cap \mathbf{W}_i = w_i(A)$ for any $j \geq i$. Now $\bigcup_{j\geq i}(w_j(A) \cap \mathbf{W}_i) = \bigcup_{j\geq i} w_i(A) = w_i(A)$. ■

Definition. Let $\langle \mathbf{W}, \circ, w\rangle$ be a $\mathrm{Tp}(m)$-quasimodel. Let $A, B \in \mathrm{Tp}$, $\alpha \in \mathbf{W}$, $\gamma \in \mathbf{W}$, and $\gamma \notin w(A\backslash B)$. We say that α is a *witness of* $\gamma \notin w(A\backslash B)$ iff $\alpha \circ \gamma$ is defined, $\alpha \in w(A)$, and $\alpha \circ \gamma \notin w(B)$.

Let $\langle \mathbf{W}, \circ, w\rangle$ be a $\mathrm{Tp}(m)$-quasimodel. Let $A, B \in \mathrm{Tp}$, $\alpha \in \mathbf{W}$, $\gamma \in \mathbf{W}$, and $\gamma \notin w(B/A)$. We say that α is a *witness of* $\gamma \notin w(B/A)$ iff $\gamma \circ \alpha$ is defined, $\alpha \in w(A)$, and $\gamma \circ \alpha \notin w(B)$.

Remark. Let $\langle \mathbf{W}, \circ, w\rangle$ be a partial semigroup model. Then for any $A \in \mathrm{Tp}$, $B \in \mathrm{Tp}$, $\gamma \in \mathbf{W}$, if $\gamma \notin w(A\backslash B)$ then there is a witness of $\gamma \notin w(A\backslash B)$ in $\langle \mathbf{W}, \circ, w\rangle$.

Definition. Let $\langle \mathbf{U}, \circ\rangle$ be a partial semigroup. Let $\mathcal{K}$ be a class of $\mathrm{Tp}(m)$-quasimodels over sub-partial-semigroups of $\langle \mathbf{U}, \circ\rangle$. We say that the class $\mathcal{K}$ is *witnessed* iff

(1) for any $\langle \mathbf{V}, \circ, v\rangle \in \mathcal{K}$, for any type of the form $A\backslash B$ from $\mathrm{Tp}(m)$, and for any $\gamma \in \mathbf{V}$, if $\gamma \notin v(A\backslash B)$ then there is a conservative extension $\langle \mathbf{W}, \circ, w\rangle$ of $\langle \mathbf{V}, \circ, v\rangle$ in $\mathcal{K}$ and $\langle \mathbf{W}, \circ, w\rangle$ contains a witness of $\gamma \notin w(A\backslash B)$;

(2) for any $\langle \mathbf{V}, \circ, v\rangle \in \mathcal{K}$, for any type of the form B/A from $\mathrm{Tp}(m)$, and for any $\gamma \in \mathbf{V}$, if $\gamma \notin v(B/A)$ then there is a conservative extension $\langle \mathbf{W}, \circ, w\rangle$ of $\langle \mathbf{V}, \circ, v\rangle$ in $\mathcal{K}$ and $\langle \mathbf{W}, \circ, w\rangle$ contains a witness of $\gamma \notin w(B/A)$.

Theorem 1 *Let m be a positive integer and $\mathcal{K}$ be a witnessed class of* $\mathrm{Tp}(m)$*-quasimodels over sub-partial-semigroups of a countable partial semigroup $\langle \mathbf{U}, \circ\rangle$. Let $E \in \mathrm{Tp}(m)$, $F \in \mathrm{Tp}(m)$, and the sequent $E\to F$ be false in a* $\mathrm{Tp}(m)$*-quasimodel from $\mathcal{K}$.*

Then $E\to F$ is also false in a partial semigroup model over a sub-partial-semigroup of $\langle \mathbf{U}, \circ\rangle$.

Proof. The following proof is similar to the R-completeness proof in [9].

Evidently there is a function $\sigma\colon \mathbf{N} \to \mathrm{Tp}(m) \times \mathbf{U}$ such that for any $\gamma \in \mathbf{U}$ and for any $C \in \mathrm{Tp}(m)$ there are infinitely many natural numbers i, for which $\sigma(i) = \langle C, \gamma\rangle$. For example, the function σ can be obtained from any bijection $\tau\colon \mathbf{N} \to \mathrm{Tp}(m) \times \mathbf{U} \times \mathbf{N}$.

Given a $\mathrm{Tp}(m)$-quasimodel $\langle \mathbf{W}_0, \circ, w_0\rangle$, in which $E\to F$ is false, we define by induction on i a conservative sequence $\langle \mathbf{W}_i, \circ, w_i\rangle$ $(i \in \mathbf{N})$.

Case 1:

If $\sigma(i) = \langle A\backslash B, \gamma\rangle$, $\gamma \in \mathbf{W}_i$, $\gamma \notin w_i(A\backslash B)$, and there are no witnesses of $\gamma \notin w_i(A\backslash B)$ in $\langle \mathbf{W}_i, \circ, w_i\rangle$, then take $\langle \mathbf{W}_{i+1}, \circ, w_{i+1}\rangle$ to be any conservative extension of $\langle \mathbf{W}_i, \circ, w_i\rangle$ in $\mathcal{K}$, containing a witness of $\gamma \notin w_{i+1}(A\backslash B)$. Such a $\mathrm{Tp}(m)$-quasimodel $\langle \mathbf{W}_{i+1}, \circ, w_{i+1}\rangle$ exists, since $\mathcal{K}$ is witnessed.

Case 2:

If $\sigma(i) = \langle B/A, \gamma\rangle$, $\gamma \in \mathbf{W}_i$, $\gamma \notin w_i(B/A)$, and there are no witnesses of $\gamma \notin w_i(B/A)$

in $\langle \mathbf{W}_i, \circ, w_i \rangle$, then take $\langle \mathbf{W}_{i+1}, \circ, w_{i+1} \rangle$ to be any conservative extension of $\langle \mathbf{W}_i, \circ, w_i \rangle$ in $\mathcal{K}$, containing a witness of $\gamma \notin w_{i+1}(B/A)$.
Case 3:
Otherwise put $\langle \mathbf{W}_{i+1}, \circ, w_{i+1} \rangle \rightleftharpoons \langle \mathbf{W}_i, \circ, w_i \rangle$.

Let $\langle \mathbf{W}_\infty, \circ, w_\infty \rangle$ be the limit of the conservative sequence $\langle \mathbf{W}_i, \circ, w_i \rangle$. Evidently, $E \to F$ is false in $\langle \mathbf{W}_\infty, \circ, w_\infty \rangle$. Now we define a valuation v over $\langle \mathbf{W}_\infty, \circ \rangle$ by induction on the complexity of a type.

$$\begin{array}{rcl} v(p_i) & \rightleftharpoons & w_\infty(p_i) \\ v(A \bullet B) & \rightleftharpoons & v(A) \circ v(B) \\ v(A \backslash B) & \rightleftharpoons & \{\gamma \mid \forall \alpha \in w(A) \text{ if } \alpha \circ \gamma \text{ is defined then } \alpha \circ \gamma \in w(B)\ \} \\ v(B/A) & \rightleftharpoons & \{\gamma \mid \forall \alpha \in w(A) \text{ if } \gamma \circ \alpha \text{ is defined then } \gamma \circ \alpha \in w(B)\ \} \end{array}$$

Evidently $\langle \mathbf{W}_\infty, \circ, v \rangle$ is a partial semigroup model. Next we verify that $w_\infty(C) = v(C)$ for any $C \in \mathrm{Tp}(m)$.

Induction on the complexity of $C \in \mathrm{Tp}(m)$. Induction step.
Case 1: $C = A \bullet B$
Obvious, since both v and w_∞ are $\mathrm{Tp}(m)$-quasimodels.
Case 2: $C = A \backslash B$
First we prove that if $\gamma \in w_\infty(A \backslash B)$ then $\gamma \in v(A \backslash B)$. Let $\gamma \in w_\infty(A \backslash B)$. Take any $\alpha \in v(A)$ such that $\alpha \circ \gamma$ is defined. By the induction hypothesis $\alpha \in w_\infty(A)$. Evidently $\alpha \circ \gamma \in \overline{w_\infty}(A(A \backslash B))$. Hence $\alpha \circ \gamma \in w_\infty(B)$ in view of $L \vdash A(A \backslash B) \to B$. By the induction hypothesis $\alpha \circ \gamma \in v(B)$. Thus $\gamma \in v(A \backslash B)$.

Now we prove that if $\gamma \notin w_\infty(A \backslash B)$ then $\gamma \notin v(A \backslash B)$. If $\gamma \notin \mathbf{W}_\infty$, then this is obvious. Let $\gamma \in \mathbf{W}_j$. There exists an integer $i \geq j$ such that $\sigma(i) = \langle A \backslash B, \gamma \rangle$. According to the construction of $\langle \mathbf{W}_{i+1}, \circ, w_{i+1} \rangle$ there exists $\alpha \in \mathbf{W}_{i+1}$ such that $\alpha \circ \gamma$ is defined, $\alpha \in w_{i+1}(A)$, and $\alpha \circ \gamma \notin w_{i+1}(B)$. Since w_∞ is conservative over w_{i+1}, we have $\alpha \in w_\infty(A)$ and $\alpha \circ \gamma \notin w_\infty(B)$. By the induction hypothesis, $\alpha \in v(A)$ and $\alpha \circ \gamma \notin v(B)$. Thus $\gamma \notin v(A \backslash B)$.
Case 3: $C = B/A$
Similar to case 2.

Now we can prove that $\langle \mathbf{W}_\infty, \circ, v \rangle$ is the desired partial semigroup model. First, $\langle \mathbf{W}_\infty, \circ \rangle$ is a sub-partial-semigroup of $\langle \mathbf{U}, \circ \rangle$. It remains to show that $v(E) \not\subseteq v(F)$.

Since $w_0(E) \not\subseteq w_0(F)$, there is $\alpha \in \mathbf{W}_0$ such that $\alpha \in w_0(E)$ and $\alpha \notin w_0(F)$. In view of Lemma 2.4 we have $\alpha \in w_\infty(E)$ and $\alpha \notin w_\infty(F)$. Thus $\alpha \in v(E)$ and $\alpha \notin v(F)$. ■

3 Faithful quasimodels over linear order

The aim of this section is to introduce "left" quasimodels $\langle \mathbf{V}^{\mathrm{lf}}_{\mathrm{m,n}}, \circ, v^{\mathrm{lf}}_{\mathrm{m,n}} \rangle$ (and "right" quasimodels $\langle \mathbf{V}^{\mathrm{rg}}_{\mathrm{m,n}}, \circ, v^{\mathrm{rg}}_{\mathrm{m,n}} \rangle$), which will later be used in the proof of Lemma 4.1, where we construct a $\mathrm{Tp}(m)$-quasimodel containing a witness for given $\delta \notin E \backslash F$ (resp. $\delta \notin F/E$).

Lemma 3.1 *There is a family of quasimodels $\langle \mathbf{V}_\Gamma, \circ, v_\Gamma \rangle$ indexed by sequences of types $\Gamma \in \mathrm{Tp}^*$, such that $\langle \mathbf{V}_\Gamma, \circ \rangle \in \mathcal{S}_{\mathbf{Z}}$ for any Γ (cf. Example 1* (f)*). We denote the domain of $\langle \mathbf{V}_\Gamma, \circ \rangle$ by $\mathbf{D}_\Gamma$ (i.e., $\mathbf{V}_\Gamma \subset \mathbf{D}_\Gamma \times \mathbf{D}_\Gamma$).*

There are designated elements $\psi \in \mathbf{D}_\Lambda$ and $\chi_\Gamma \in \mathbf{D}_\Gamma$ such that

(i) $(\forall \Gamma \in \mathrm{Tp}^*)\ (\forall C \in \mathrm{Tp})\ \langle \psi, \chi_\Gamma \rangle \in v_\Gamma(C) \Leftrightarrow L \vdash \Gamma \to C$

(ii) $(\forall \Gamma \in \mathrm{Tp}^*)\ (\forall \Pi \in \mathrm{Tp}^*)\ \mathbf{D}_\Gamma \subseteq \mathbf{D}_{\Gamma\Pi}$ *and* $\mathbf{V}_\Gamma \subseteq \mathbf{V}_{\Gamma\Pi}$

(iii) $(\forall \Gamma \in \mathrm{Tp}^*)\ (\forall \Pi \in \mathrm{Tp}^*)\ (\forall C \in \mathrm{Tp})\ v_\Gamma(C) \subseteq v_{\Gamma\Pi}(C)$

(iv) $(\forall \Gamma \in \mathrm{Tp}^*)\ (\forall B \in \mathrm{Tp})\ \langle \chi_\Gamma, \chi_{\Gamma B} \rangle \in v_{\Gamma B}(B)$

Lemma 3.1 will be proved in Section 6.5.

Lemma 3.2 *If $L \not\vdash E \to F$ then there is a quasimodel $\langle \mathbf{W}, \circ, w \rangle$ such that $\langle \mathbf{W}, \circ \rangle \in \mathcal{S}_{\mathbf{Z}}$ and $w(E) \not\subseteq w(F)$.*

Proof. Consider $\langle \mathbf{V}_E, \circ, v_E \rangle$. In view of $L \not\vdash E \to F$ we have $\langle \psi, \chi_E \rangle \notin v_E(F)$. In view of $L \vdash E \to E$ we have $\langle \psi, \chi_E \rangle \in v_E(E)$. ■

Lemma 3.3 *There is an R-quasimodel $\langle \mathbf{V}^{\mathrm{lf}}, \circ, v^{\mathrm{lf}} \rangle$ over an infinite linear order $\mathbf{V}^{\mathrm{lf}} \subset \mathbf{D}^{\mathrm{lf}} \times \mathbf{D}^{\mathrm{lf}}$, there is a designated element $g \in \mathbf{D}^{\mathrm{lf}}$, and there is a family of elements $h_\Gamma \in \mathbf{D}^{\mathrm{lf}}$ for $\Gamma \in \mathrm{Tp}^*$, such that*

(i) $(\forall \Gamma \in \mathrm{Tp}^*)\ (\forall C \in \mathrm{Tp})\ \langle g, h_\Gamma \rangle \in v^{\mathrm{lf}}(C) \Leftrightarrow L \vdash \Gamma \to C$

(ii) $(\forall \Gamma \in \mathrm{Tp}^*)\ (\forall B \in \mathrm{Tp})\ \langle h_\Gamma, h_{\Gamma B} \rangle \in v^{\mathrm{lf}}(B)$

Proof. We construct the quasimodel $\langle \mathbf{V}^{\mathrm{lf}}, \circ, v^{\mathrm{lf}} \rangle$ using the family of quasimodels $\langle \mathbf{V}_\Gamma, \circ, v_\Gamma \rangle$ from Lemma 3.1.

$$\mathbf{D}^{\mathrm{lf}} \rightleftharpoons \{\xi_\Gamma^s \mid \Gamma \in \mathrm{Tp}^*, \quad s \in \mathbf{D}_\Gamma\}, \text{ where } \xi_\Gamma^s \text{ are new formal symbols.}$$

$$\mathbf{W} \rightleftharpoons \{\langle \xi_\Gamma^s, \xi_{\Gamma\Delta}^t \rangle \mid \Gamma \in \mathrm{Tp}^*,\ \Delta \in \mathrm{Tp}^*,\ s \in \mathbf{D}_\Gamma,\ t \in \mathbf{D}_{\Gamma\Delta},\ \langle s, t \rangle \in \mathbf{V}_{\Gamma\Delta}\}$$

Evidently $\mathbf{W}$ is irreflexive. Next we verify that $\mathbf{W}$ is transitive.

Let $\langle \xi_\Gamma^r, \xi_{\Gamma\Delta}^s \rangle \in \mathbf{W}$ and $\langle \xi_{\Gamma\Delta}^s, \xi_{\Gamma\Delta\Pi}^t \rangle \in \mathbf{W}$. Then $\langle r, s \rangle \in \mathbf{V}_{\Gamma\Delta}$ and $\langle s, t \rangle \in \mathbf{V}_{\Gamma\Delta\Pi}$. From Lemma 3.1 (ii) we obtain $\langle r, s \rangle \in \mathbf{V}_{\Gamma\Delta\Pi}$. Thus $\langle r, t \rangle = \langle r, s \rangle \circ \langle s, t \rangle \in \mathbf{V}_{\Gamma\Delta\Pi}$. Hence $\langle \xi_\Gamma^r, \xi_{\Gamma\Delta\Pi}^t \rangle \in \mathbf{W}$.

We take $\mathbf{V}^{\mathrm{lf}}$ to be any linear order on $\mathbf{D}^{\mathrm{lf}}$ such that $\mathbf{W} \subseteq \mathbf{V}^{\mathrm{lf}}$. We put

$$v^{\mathrm{lf}}(A) \rightleftharpoons \{\langle \xi_\Gamma^s, \xi_{\Gamma\Delta}^t \rangle \mid \Gamma \in \mathrm{Tp}^*,\ \Delta \in \mathrm{Tp}^*,\ s \in \mathbf{D}_\Gamma,\ t \in \mathbf{D}_{\Gamma\Delta},\ \langle s, t \rangle \in v_{\Gamma\Delta}(A)\}.$$

$$g \rightleftharpoons \xi_\Lambda^\psi \qquad h_\Gamma \rightleftharpoons \xi_\Gamma^{\chi_\Gamma}$$

First, we verify that $\langle \mathbf{V}^{\mathrm{lf}}, \circ, v^{\mathrm{lf}} \rangle$ is a quasimodel (conditions (2) and (1) from the definition of a quasimodel at page 5).

(2)

If $L \vdash A \rightarrow B$, then $v_{\Gamma\Delta}(A) \subseteq v_{\Gamma\Delta}(B)$ for any $\Gamma, \Delta \in \mathrm{Tp}^*$, whence $v^{\mathrm{lf}}(A) \subseteq v^{\mathrm{lf}}(B)$.

(1) $v^{\mathrm{lf}}(A \bullet B) \subseteq v^{\mathrm{lf}}(A) \circ v^{\mathrm{lf}}(B)$

Let $\langle \xi^r_\Gamma, \xi^t_{\Gamma\Delta} \rangle \in v^{\mathrm{lf}}(A \bullet B)$. Since $\langle r, t \rangle \in v_{\Gamma\Delta}(A \bullet B) = v_{\Gamma\Delta}(A) \circ v_{\Gamma\Delta}(B)$, there is s such that $\langle r, s \rangle \in v_{\Gamma\Delta}(A)$ and $\langle s, t \rangle \in v_{\Gamma\Delta}(B)$. Note that $s \in \mathbf{D}_{\Gamma\Delta}$, since $\langle s, t \rangle \in v_{\Gamma\Delta}(B) \subset \mathbf{D}_{\Gamma\Delta} \times \mathbf{D}_{\Gamma\Delta}$.

According to the definition of v^{lf}, $\langle \xi^r_\Gamma, \xi^s_{\Gamma\Delta} \rangle \in v^{\mathrm{lf}}(A)$ and $\langle \xi^s_{\Gamma\Delta}, \xi^t_{\Gamma\Delta} \rangle \in v^{\mathrm{lf}}(B)$. Thus $\langle \xi^r_\Gamma, \xi^t_{\Gamma\Delta} \rangle \in v^{\mathrm{lf}}(A) \circ v^{\mathrm{lf}}(B)$.

(1) $v^{\mathrm{lf}}(A) \circ v^{\mathrm{lf}}(B) \subseteq v^{\mathrm{lf}}(A \bullet B)$

Let $\langle \xi^r_\Gamma, \xi^s_{\Gamma\Delta} \rangle \in v^{\mathrm{lf}}(A)$ and $\langle \xi^s_{\Gamma\Delta}, \xi^t_{\Gamma\Delta\Pi} \rangle \in v^{\mathrm{lf}}(B)$. We have $\langle r, s \rangle \in v_{\Gamma\Delta}(A)$ and $\langle s, t \rangle \in v_{\Gamma\Delta\Pi}(B)$. By Lemma 3.1 (iii), $v_{\Gamma\Delta}(A) \subseteq v_{\Gamma\Delta\Pi}(A)$. Thus $\langle r, s \rangle \in v_{\Gamma\Delta\Pi}(A)$, whence $\langle r, t \rangle = \langle r, s \rangle \circ \langle s, t \rangle \in v_{\Gamma\Delta\Pi}(A) \circ v_{\Gamma\Delta\Pi}(B) = v_{\Gamma\Delta\Pi}(A \bullet B)$. We see that $\langle \xi^r_\Gamma, \xi^t_{\Gamma\Delta\Pi} \rangle \in v^{\mathrm{lf}}(A \bullet B)$.

Next, we prove that $\langle \mathbf{V}^{\mathrm{lf}}, \circ, v^{\mathrm{lf}} \rangle$ has the properties (i) and (ii).

(i)

Evidently, $\langle g, h_\Gamma \rangle \in v^{\mathrm{lf}}(C)$ if and only if $\langle \psi, \chi_\Gamma \rangle \in v_\Gamma(C)$. According to Lemma 3.1 (i), $\langle \psi, \chi_\Gamma \rangle \in v_\Gamma(C)$ if and only if $L \vdash \Gamma \rightarrow C$.

(ii)

By Lemma 3.1 (iv), $\langle \chi_\Gamma, \chi_{\Gamma B} \rangle \in v_{\Gamma B}(B)$, whence $\langle \xi^{\chi_\Gamma}_\Gamma, \xi^{\chi_{\Gamma B}}_{\Gamma B} \rangle \in v^{\mathrm{lf}}(B)$. ■

Lemma 3.4 *For any positive integers m and n there is a* $\mathrm{Tp}(m)$*-quasimodel* $\langle \mathbf{V}^{\mathrm{lf}}_{\mathrm{m,n}}, \circ, v^{\mathrm{lf}}_{\mathrm{m,n}} \rangle$ *over a finite linear order* $\mathbf{V}^{\mathrm{lf}}_{\mathrm{m,n}} \subset \mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}} \times \mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}}$*, there is a designated element* $g \in \mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}}$*, and there is a family of elements* $h_\Gamma \in \mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}}$ *for* $\Gamma \in \mathrm{LST}_{m,n}$*, such that*

(i) $(\forall \Gamma \in \mathrm{LST}_{m,n})\ (\forall C \in \mathrm{Tp}(m))\ \langle g, h_\Gamma \rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(C) \Leftrightarrow L \vdash \Gamma \rightarrow C$

(ii) $(\forall \Gamma \in \mathrm{LST}_{m,n-1})\ (\forall B \in \mathrm{Tp}(m))\ \langle h_\Gamma, h_{\Gamma B} \rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(B)$

Proof.

$$\mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}} \rightleftharpoons \{\xi^s_\Gamma \mid \Gamma \in \mathrm{LST}_{m,n},\ s \in \mathbf{D}_\Gamma\} \cup \{\xi^s_\Lambda \mid s \in \mathbf{D}_\Lambda\}$$

$$\mathbf{V}^{\mathrm{lf}}_{\mathrm{m,n}} \rightleftharpoons \mathbf{V}^{\mathrm{lf}} \cap (\mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}} \times \mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}})$$

$$v^{\mathrm{lf}}_{\mathrm{m,n}}(A) \rightleftharpoons v^{\mathrm{lf}}(A) \cap (\mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}} \times \mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}}) \ \text{ for any } A \in \mathrm{Tp}(m)$$

$$g \rightleftharpoons \xi^\psi_\Lambda \qquad h_\Gamma \rightleftharpoons \xi^{\chi_\Gamma}_\Gamma$$

It remains to repeat the proof of Lemma 3.3. ■

Evidently, all the lemmas of this section have also inverted duals. We formulate the dual of Lemma 3.4.

Lemma 3.5 *For any positive integers m and n there is a* $\mathrm{Tp}(m)$*-quasimodel* $\langle \mathbf{V}^{\mathrm{rg}}_{\mathrm{m,n}}, \circ, v^{\mathrm{rg}}_{\mathrm{m,n}} \rangle$ *over a finite linear order* $\mathbf{V}^{\mathrm{rg}}_{\mathrm{m,n}} \subset \mathbf{D}^{\mathrm{rg}}_{\mathrm{m,n}} \times \mathbf{D}^{\mathrm{rg}}_{\mathrm{m,n}}$*, there is a designated element* $g \in \mathbf{D}^{\mathrm{rg}}_{\mathrm{m,n}}$*, and there is a family of elements* $h_\Gamma \in \mathbf{D}^{\mathrm{rg}}_{\mathrm{m,n}}$ *for* $\Gamma \in \mathrm{LST}_{m,n}$*, such that*

(i) $(\forall \Gamma \in \mathrm{LST}_{m,n})\ (\forall C \in \mathrm{Tp}(m))\ \langle h_\Gamma, g \rangle \in v^{\mathrm{rg}}_{\mathrm{m,n}}(C) \Leftrightarrow L \vdash \Gamma \rightarrow C$

(ii) $(\forall \Gamma \in \mathrm{LST}_{m,n-1})\ (\forall B \in \mathrm{Tp}(m))\ \langle h_{B\Gamma}, h_\Gamma \rangle \in v^{\mathrm{rg}}_{\mathrm{m,n}}(B)$

4 R-completeness

In this section we demonstrate how a partial semigroup $\langle \mathbf{W}, \circ \rangle$ and a Tp(m)-quasimodel $\langle \mathbf{V}, \circ, v \rangle$ satisfying certain conditions can be used to construct an 'almost' Tp(m)-quasimodel $\langle \mathbf{W}, \circ, u \rangle$, which is 'conservative' over $\langle \mathbf{V}, \circ, v \rangle$ (cf. Lemma 4.1).

Using this result, we are going to prove that the class of all Tp(m)-quasimodels over finite intervals of integers is witnessed (cf. Lemma 4.2) and the class of certain Tp(m)-quasimodels over finitely generated free semigroups is witnessed (cf. Lemma 5.1).

Lemma 4.1 *Given $m, n \in \mathbf{N}$, let $\mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}} = [0, k]$ (and thus $\mathbf{V}^{\mathrm{lf}}_{\mathrm{m,n}} = \{\langle s, t \rangle \mid 0 \le s < t \le k\}$). Let $\langle \mathbf{V}, \circ, v \rangle$ be a* Tp(m)*-quasimodel, $\langle \mathbf{W}, \circ \rangle$ be a partial semigroup, $E \in \mathrm{Tp}(m)$, $\mathcal{R} \subseteq \mathbf{V}$, $\mathcal{T} \subseteq \mathbf{W}$, and π be a function $\pi\colon \mathbf{V}^{\mathrm{lf}}_{\mathrm{m,n}} \to \mathbf{W}$. We denote*

$$\begin{aligned} \mathcal{P}_0 &= \{\pi\langle s, t \rangle \mid 0 \le s < t < k\}; \\ \mathcal{P}_1 &= \{\pi\langle s, k \rangle \mid 0 \le s < k\}; \\ \mathcal{P}_2 &= \mathcal{P}_1 \circ \mathcal{R}; \\ \mathcal{P} &= \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2. \end{aligned}$$

Let the following conditions hold.

(1) *The partial semigroup $\langle \mathbf{V}, \circ \rangle$ is a sub-partial-semigroup of $\langle \mathbf{W}, \circ \rangle$.*

(2) $\mathcal{P}_0 \cap \mathcal{P}_1 = \emptyset$

(3) $\mathcal{P}_0 \cap \mathcal{P}_2 = \emptyset$

(4) $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$

(5) $\mathcal{P} \cap \mathbf{V} = \emptyset$

(6) $\mathcal{P} \cap \mathcal{T} = \emptyset$

(7) $\mathcal{T} \cap \mathbf{V} = \emptyset$

(8) $\mathcal{T} \circ (\mathcal{P} \cup \mathbf{V}) \subseteq \mathcal{T} \qquad (\mathcal{P} \cup \mathbf{V}) \circ \mathcal{T} \subseteq \mathcal{T}$

(9) $\pi\langle r, s \rangle \circ \pi\langle s, t \rangle = \pi\langle r, t \rangle$

(10) *If $s \ne s'$ and $\pi\langle r, s \rangle \circ \pi\langle s', t \rangle$ is defined, then $\pi\langle r, s \rangle \circ \pi\langle s', t \rangle \in \mathcal{T}$.*

(11) $\mathbf{V} \circ \mathcal{P} \subseteq \mathcal{T}$

(12) $\mathcal{P}_0 \circ \mathbf{V} \subseteq \mathcal{T}$

(13) *For any $s < k$ and $\rho \in \mathcal{R}$, $\pi\langle s, k \rangle \circ \rho$ is defined.*

(14) $\mathcal{P}_1 \circ \{\beta \in \mathbf{W} \mid \beta \notin \mathcal{R}\} \subseteq \mathcal{T}$

(15) *If* $s < k$, $s' < k$, *and* $\pi\langle s,k\rangle = \pi\langle s',k\rangle$ *then* $s = s'$.

(16) *If* $s < k$, $s' < k$, $\rho \in \mathcal{R}$, $\rho' \in \mathcal{R}$, *and* $\pi\langle s,k\rangle \circ \rho = \pi\langle s',k\rangle \circ \rho'$ *then* $s = s'$ *and* $\rho = \rho'$.

(17) $\alpha_1 \circ \ldots \circ \alpha_n \notin \mathcal{R}$, *for any* $\alpha_1, \ldots, \alpha_n \in \mathbf{W}$. *(Here* n *is the given natural number.)*

Then there is a function $u\colon \mathrm{Tp}(m) \to \mathbf{P}(\mathbf{W})$ *satisfying the following conditions* (i)–(vii).

(i) *For any* $A{\bullet}B \in \mathrm{Tp}(m)$, $u(A{\bullet}B) \subseteq u(A) \circ u(B)$.

(ii) *For any* $B_1, \ldots, B_l, C \in \mathrm{Tp}(m)$, *if* $L \vdash B_1 \ldots B_l \to C$, *then* $u(B_1) \circ \ldots \circ u(B_l) \subseteq u(C) \cup \mathcal{T}$.

(iii) $\pi\langle g,k\rangle \in u(E)$ *(Recall that* g *is the designated element in* $\mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}}$ *and thus* $0 \le g \le k$.*)*

(iv) *If* $F \in \mathrm{Tp}(m)$ *and* $L \not\vdash E \to F$, *then* $\pi\langle g,k\rangle \notin u(F)$.

(v) *If* $F \in \mathrm{Tp}(m)$, $\rho \in \mathcal{R}$, *and* $\rho \notin v(E\backslash F)$, *then* $\pi\langle g,k\rangle \circ \rho \notin u(F)$.

(vi) $v(A) \subseteq u(A)$ *for any* $A \in \mathrm{Tp}(m)$.

(vii) $u(A) \subseteq v(A) \cup \mathcal{P}$ *for any* $A \in \mathrm{Tp}(m)$.

Proof. We define the function u associating subsets of $\mathbf{W}$ not only with single types from $\mathrm{Tp}(m)$, but also with sequences of types from $\mathrm{Tp}(m)$, i.e., $u\colon \mathrm{Tp}(m)^+ \to \mathbf{P}(\mathbf{W})$.

$$\begin{array}{rcl}
u_0(\Theta) & \rightleftharpoons & \{\pi\langle s,t\rangle \mid 0 \le s < t < k \ \text{ and } \ \langle s,t\rangle \in \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta)\} \\
u_1(\Theta) & \rightleftharpoons & \{\pi\langle s,k\rangle \mid 0 \le s < k \ \text{ and } \ \langle s,h_E\rangle \in \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta)\} \\
u_2(\Theta) & \rightleftharpoons & \{\pi\langle s,k\rangle \circ \rho \mid 0 \le s < k,\ \rho \in \mathcal{R}, \ \text{ and } \ \exists \Delta \in \mathrm{LST}_{m,n-1}, \\
 & & \qquad \rho \in \overline{v}(\Delta),\ \langle s,h_{E\Delta}\rangle \in \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta)\} \\
u(\Theta) & \rightleftharpoons & u_0(\Theta) \cup u_1(\Theta) \cup u_2(\Theta) \cup \overline{v}(\Theta)
\end{array}$$

Note that $u_0(\Theta) \subseteq \mathcal{P}_0$, $u_1(\Theta) \subseteq \mathcal{P}_1$, $u_2(\Theta) \subseteq \mathcal{P}_2$, and $u(\Theta) \subseteq \mathcal{P} \cup \mathbf{V}$.

Lemma 4.1.1 *Let* $\Theta \in \mathrm{Tp}(m)^+$ *and* $B \in \mathrm{Tp}(m)$. *Then* $u(\Theta) \circ u(B) \subseteq u(\Theta B) \cup \mathcal{T}$.

Proof. Let $\gamma \in u(\Theta) \circ u(B)$. Then $\gamma = \alpha \circ \beta$ for some $\alpha \in u(\Theta) = u_0(\Theta) \cup u_1(\Theta) \cup u_2(\Theta) \cup \overline{v}(\Theta)$ and $\beta \in u(B) = u_0(B) \cup u_1(B) \cup u_2(B) \cup v(B)$. We consider the corresponding sixteen cases and prove that $\alpha \circ \beta \in u_0(\Theta B) \cup u_1(\Theta B) \cup u_2(\Theta B) \cup v(\Theta B) \cup \mathcal{T}$.

Case 1: $\alpha \in u_0(\Theta)$
$\alpha = \pi\langle r,s\rangle$, $0 \le r < s < k$, $\langle r,s\rangle \in \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta)$
Case 1a: $\beta \in u_0(B)$
$\beta = \pi\langle s',t\rangle$, $0 \le s' < t < k$, $\langle s',t\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(B)$
If $s \ne s'$, then $\alpha \circ \beta \in \mathcal{T}$ in view of (10).

If $s = s'$, then $\alpha \circ \beta = \pi\langle r,s\rangle \circ \pi\langle s,t\rangle = \pi\langle r,t\rangle \in u_0(\Theta B)$, since $\langle r,t\rangle = \langle r,s\rangle \circ \langle s,t\rangle \in \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta) \circ v^{\mathrm{lf}}_{\mathrm{m,n}}(B) = \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta B)$.

Case 1b: $\beta \in u_1(B)$

$\beta = \pi\langle s',k\rangle$, $\langle s',h_E\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(B)$

If $s \neq s'$, then $\alpha \circ \beta \in \mathcal{T}$ in view of (10).

If $s = s'$, then $\alpha \circ \beta = \pi\langle r,s\rangle \circ \pi\langle s,k\rangle = \pi\langle r,k\rangle \in u_1(\Theta B)$, since $\langle r,h_E\rangle = \langle r,s\rangle \circ \langle s,h_E\rangle \in \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta) \circ v^{\mathrm{lf}}_{\mathrm{m,n}}(B) = \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta B)$.

Case 1c: $\beta \in u_2(B)$

$\beta = \pi\langle s',k\rangle \circ \rho$, $\rho \in \mathcal{R}$, $\Delta \in \mathrm{LST}_{m,n-1}$, $\rho \in \overline{v}(\Delta)$, $\langle s',h_{E\Delta}\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(B)$, $0 \leq s' < k$

If $s \neq s'$, then $\alpha \circ \beta = \pi\langle r,s\rangle \circ \pi\langle s',k\rangle \circ \rho \in \mathcal{T} \circ \rho \subseteq \mathcal{T} \circ \mathbf{V} \subseteq \mathcal{T}$.

If $s = s'$, then $\alpha \circ \beta = \pi\langle r,k\rangle \circ \rho \in u_2(\Theta B)$, since $\langle r,h_{E\Delta}\rangle \in \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta) \circ v^{\mathrm{lf}}_{\mathrm{m,n}}(B) = \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta B)$.

Case 1d: $\beta \in v(B)$

Evidently $\alpha \circ \beta \in \pi\langle r,s\rangle \circ v(B) \subseteq \mathcal{P}_0 \circ \mathbf{V} \subseteq \mathcal{T}$ in view of (12).

Case 2: $\alpha \in u_1(\Theta)$

$\alpha = \pi\langle r,k\rangle$, $\langle r,h_E\rangle \in \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta)$, $0 \leq r < k$

Case 2ab: $\beta \in u_0(B) \cup u_1(B)$

$\beta = \pi\langle s,t\rangle$, $0 \leq s < t \leq k$

From (10) we obtain $\alpha \circ \beta = \pi\langle r,k\rangle \circ \pi\langle s,t\rangle \in \mathcal{T}$, since $k \neq s$.

Case 2c: $\beta \in u_2(B)$

From (10) and (8) we obtain $\alpha \circ \beta \in \mathcal{T} \circ \mathbf{V} \subseteq \mathcal{T}$ (cf. case 2ab and case 1c).

Case 2d: $\beta \in v(B)$

If $\beta \notin \mathcal{R}$, then $\alpha \circ \beta \in \mathcal{T}$ in view of (14).

Now we prove that if $\beta \in \mathcal{R}$ then $\alpha \circ \beta \in u_2(\Theta B)$. We take $\Delta = B$ and $\rho = \beta$. From (17) we see that $n > 1$. Thus $\Delta \in \mathrm{LST}_{m,n-1}$. By Lemma 3.4 (ii) we have $\langle h_E, h_{EB}\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(B)$. Thus $\langle r,h_{EB}\rangle = \langle r,h_E\rangle \circ \langle h_E,h_{EB}\rangle \in \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta) \circ v^{\mathrm{lf}}_{\mathrm{m,n}}(B) = \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta B)$, whence $\alpha \circ \beta \in u_2(\Theta B)$.

Case 3: $\alpha \in u_2(\Theta)$

$\alpha = \pi\langle r,k\rangle \circ \rho$, $\rho \in \mathcal{R}$, $\Delta \in \mathrm{LST}_{m,n-1}$, $\Delta \neq \Lambda$, $\rho \in \overline{v}(\Delta)$, $\langle r,h_{E\Delta}\rangle \in \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta)$, $0 \leq r < k$

Note that $\alpha \in \mathcal{P} \circ \mathbf{V}$.

Case 3abc: $\beta \in u_0(B) \cup u_1(B) \cup u_2(B) \subseteq \mathcal{P}$

In view of (11) and (8) we have $\alpha \circ \beta \in \mathcal{P} \circ \mathbf{V} \circ \mathcal{P} \subseteq \mathcal{P} \circ \mathcal{T} \subseteq \mathcal{T}$.

Case 3d: $\beta \in v(B)$

If $\rho \circ \beta \notin \mathcal{R}$, then $\alpha \circ \beta = \pi\langle r,k\rangle \circ (\rho \circ \beta) \in \mathcal{T}$ in view of (14).

Now we prove that if $\rho \circ \beta \in \mathcal{R}$ then $\alpha \circ \beta \in u_2(\Theta B)$. We take $\Delta' = \Delta B$ and $\rho' = \rho \circ \beta$. Evidently $\rho \circ \beta \in \overline{v}(\Delta) \circ v(B) = \overline{v}(\Delta B)$. Thus $\rho \circ \beta = \alpha_1 \circ \ldots \circ \alpha_l$, where l is the number of types in the sequence ΔB. In view of (17), $\Delta B \in \mathrm{LST}_{m,n-1}$. By Lemma 3.4 (ii) we have $\langle h_{E\Delta}, h_{E\Delta B}\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(B)$. Thus $\langle r,h_{E\Delta B}\rangle = \langle r,h_{E\Delta}\rangle \circ \langle h_{E\Delta},h_{E\Delta B}\rangle \in \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta) \circ v^{\mathrm{lf}}_{\mathrm{m,n}}(B) = \overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(\Theta B)$.

Case 4: $\alpha \in \overline{v}(\Theta)$

Case 4abc: $\beta \in u_0(B) \cup u_1(B) \cup u_2(B) \subseteq \mathcal{P}$

In view of (11) we have $\alpha \circ \beta \in \mathbf{V} \circ \mathcal{P} \subseteq \mathcal{T}$.

Case 4d: $\beta \in v(B)$
Evidently $\alpha \circ \beta \in \overline{v}(\Theta) \circ v(B) = \overline{v}(\Theta B)$. ■

Lemma 4.1.2 *Let $B_1 \in \mathrm{Tp}(m)$, ..., $B_l \in \mathrm{Tp}(m)$. Then $u(B_1)\circ \ldots \circ u(B_l) \subseteq u(B_1 \ldots B_l) \cup \mathcal{T}$.*

Proof. Induction on l. Induction step. We must prove that if $u(B_1) \circ \ldots \circ u(B_l) \subseteq u(B_1 \ldots B_l) \cup \mathcal{T}$ then $u(B_1) \circ \ldots \circ u(B_l) \circ u(B_{l+1}) \subseteq u(B_1 \ldots B_l B_{l+1}) \cup \mathcal{T}$. It is sufficient to verify that $(u(B_1 \ldots B_l) \cup \mathcal{T}) \circ u(B_{l+1}) \subseteq u(B_1 \ldots B_l B_{l+1}) \cup \mathcal{T}$.

From Lemma 4.1.1 we obtain $u(B_1 \ldots B_l) \circ u(B_{l+1}) \subseteq u(B_1 \ldots B_l B_{l+1}) \cup \mathcal{T}$. According to (8), $\mathcal{T} \circ u(B_{l+1}) \subseteq u(B_1 \ldots B_l B_{l+1}) \cup \mathcal{T}$. ■

Lemma 4.1.3 *Let $A{\bullet}B \in \mathrm{Tp}(m)$ and $\gamma \in u(A{\bullet}B)$. Then there are $\alpha \in u(A)$ and $\beta \in u(B)$ such that $\alpha \circ \beta = \gamma$.*

Proof.
Case 1: $\gamma \in u_0(A{\bullet}B)$
$\gamma = \pi\langle r, t\rangle$, $0 \le r < t < k$, $\langle r, t\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(A{\bullet}B)$
Since $\langle \mathbf{V}^{\mathrm{lf}}_{\mathrm{m,n}}, \circ, v^{\mathrm{lf}}_{\mathrm{m,n}}\rangle$ is a $\mathrm{Tp}(m)$-quasimodel, there is $s \in [0, k]$ such that $\langle r, s\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(A)$ and $\langle s, t\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(B)$.

Now $\gamma = \pi\langle r, t\rangle = \pi\langle r, s\rangle \circ \pi\langle s, t\rangle \in u_0(A) \circ u_0(B)$.
Case 2: $\gamma \in u_1(A{\bullet}B)$
$\gamma = \pi\langle r, k\rangle$, $\langle r, h_E\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(A{\bullet}B)$, $0 \le r < k$
Like in case 1, there is $s \in [0, k]$ such that $\langle r, s\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(A)$ and $\langle s, h_E\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(B)$.

Thus $\gamma = \pi\langle r, k\rangle = \pi\langle r, s\rangle \circ \pi\langle s, k\rangle \in u_0(A) \circ u_1(B)$.
Case 3: $\gamma \in u_2(A{\bullet}B)$
$\gamma = \pi\langle r, k\rangle \circ \rho$, $\rho \in \mathcal{R}$, $\Delta \in \mathrm{LST}_{m,n-1}$, $\rho \in \overline{v}(\Delta)$, $\langle r, h_{E\Delta}\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(A{\bullet}B)$, $0 \le r < k$
There is s such that $\langle r, s\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(A)$ and $\langle s, h_{E\Delta}\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(B)$.

Now $\pi\langle r, s\rangle \in u_0(A)$ and $\pi\langle s, k\rangle \circ \rho \in u_2(B)$, whence $\gamma \in u_0(A) \circ u_2(B)$.
Case 4: $\gamma \in v(A{\bullet}B)$
Obvious from $v(A{\bullet}B) \subseteq v(A) \circ v(B)$. ■

We continue the proof of Lemma 4.1.

(i)
See Lemma 4.1.3.

(ii)
Let $B_1, \ldots, B_l, C \in \mathrm{Tp}(m)$ and $L \vdash B_1 \ldots B_l {\to} C$. According to Lemma 4.1.2, $u(B_1) \circ \ldots \circ u(B_l) \subseteq u(B_1 \ldots B_l) \cup \mathcal{T}$. It remains to prove that $u(B_1 \ldots B_l) \subseteq u(C)$. This follows from $\overline{v}(B_1 \ldots B_l) \subseteq v(C)$ and $\overline{v^{\mathrm{lf}}_{\mathrm{m,n}}}(B_1 \ldots B_l) \subseteq v^{\mathrm{lf}}_{\mathrm{m,n}}(C)$.

(iii)
From Lemma 3.4 (i) we obtain $\langle g, h_E\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(E)$. Thus $\pi\langle g, k\rangle \in u_1(E)$.

(iv)
Let $F \in \mathrm{Tp}(m)$ and $\pi\langle g, k\rangle \in u(F)$. Evidently $\pi\langle g, k\rangle \in \mathcal{P}_1$. From (2), (4), and (5) we see that $\pi\langle g, k\rangle \in u_1(F)$. Thus $\langle g, h_E\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(F)$ according to (15). From Lemma 3.4 (i) we obtain $L \vdash E {\to} F$.

(v)
Let $F \in \mathrm{Tp}(m)$, $\rho \in \mathcal{R}$, and $\pi\langle g,k\rangle \circ \rho \in u(F)$. Evidently $\pi\langle g,k\rangle \circ \rho \in \mathcal{P}_2$. From (3), (4), and (5) we see that $\pi\langle g,k\rangle \circ \rho \in u_2(F)$. According to (16) there is $\Delta \in \mathrm{LST}_{m,n-1}$ such that $\rho \in \overline{v}(\Delta)$ and $\langle g, h_{E\Delta}\rangle \in v^{\mathrm{lf}}_{\mathrm{m,n}}(F)$. From Lemma 3.4 (i) we obtain $L \vdash E\Delta \to F$. Applying the rule $(\to\backslash)$ we derive $L \vdash \Delta \to E\backslash F$, whence $\overline{v}(\Delta) \subseteq v(E\backslash F)$. We have proved that $\rho \in v(E\backslash F)$.

(vi) Obvious.

(vii) Obvious. ■

Definition. By $\mathcal{K}^m_{\mathbf{Z}}$ we denote the class of all $\mathrm{Tp}(m)$-quasimodels over binary relational frames from $\mathcal{S}_{\mathbf{Z}}$ (cf. Example 1 (f)).

Lemma 4.2 *Let $\langle \mathbf{V}, \circ, v\rangle \in \mathcal{K}^m_{\mathbf{Z}}$, $E\backslash F \in \mathrm{Tp}(m)$, $\delta \in \mathbf{V}$, and $\delta \notin v(E\backslash F)$. Then there is $\langle \mathbf{W}, \circ, w\rangle \in \mathcal{K}^m_{\mathbf{Z}}$ such that $\langle \mathbf{W}, \circ, w\rangle$ is a conservative extension of $\langle \mathbf{V}, \circ, v\rangle$ and $\langle \mathbf{W}, \circ, w\rangle$ contains a witness of $\delta \notin w(E\backslash F)$ (i.e., there is $\alpha \in w(E)$ such that $\alpha \circ \delta \notin w(F)$).*

Proof. Let $\langle \mathbf{V}, \circ\rangle = \langle \mathbf{W}_{[p,q]}, \circ\rangle \in \mathcal{S}_{\mathbf{Z}}$ and $\delta = \langle a,b\rangle \in \mathbf{V}$ (i.e., $p \le a < b \le q$). Put $\mathcal{T} \rightleftharpoons \emptyset$, $\mathcal{R} \rightleftharpoons \{\langle a,j\rangle \mid a < j \le q\}$, and $n \rightleftharpoons q-a+1$.

Recall that we identify $\mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}}$ with $[0,k]$ for a suitable natural number k. We take $\langle \mathbf{W}, \circ\rangle$ to be $\langle \mathbf{W}_{[p-k,q]}, \circ\rangle$.

We define

$$\begin{aligned} \pi\langle s,k\rangle &\rightleftharpoons \langle p-k+s, a\rangle; \\ \pi\langle s,t\rangle &\rightleftharpoons \langle p-k+s, p-k+t\rangle \text{ if } t<k. \end{aligned}$$

Note that

$$\begin{aligned} \mathcal{P}_0 &= \{\langle i,j\rangle \mid p-k \le i < j < p\}; \\ \mathcal{P}_1 &= \{\langle i,a\rangle \mid p-k \le i < p\}; \\ \mathcal{P}_2 &= \{\langle i,j\rangle \mid p-k \le i < p,\ a < j \le q\}. \end{aligned}$$

The conditions (1)–(17) from Lemma 4.1 are easy to verify. We take w to be the function u from Lemma 4.1.

According to Lemma 4.1 (i) and (ii), $\langle \mathbf{W}, \circ, w\rangle$ is a $\mathrm{Tp}(m)$-quasimodel. The conservativity of $\langle \mathbf{W}, \circ, w\rangle$ over $\langle \mathbf{V}, \circ, v\rangle$ follows from Lemma 4.1 (vi) and (vii). The witness of $\delta \notin w(E\backslash F)$ is $\pi\langle g,k\rangle$ (cf. Lemma 4.1 (iii) and (v), note that $\delta \in \mathcal{R}$). ■

Theorem 2 *The Lambek calculus is complete with respect to the class of all R-models on sub-partial-semigroups of $\langle <_{\mathbf{Z}}, \circ\rangle$ (cf. Example 1 (e)).*

Proof. Immediate from Theorem 1, Lemma 3.2, and Lemma 4.2. ■

Remark. The Lambek calculus is also complete with respect to the class of all R-models on the partial semigroup $\langle <_{\mathbf{Z}}, \circ\rangle$ itself.

Open question. Is the Lambek calculus complete w.r.t. finite R-models? In particular, does the proof of Theorem 2 give a finite countermodel for any given underivable sequent?

5 L-completeness

Definition. Let $\mathcal{V}$ be any alphabet, i.e., any set, the elements of which are called symbols. We denote by $\mathcal{V}^+$ the set of all non-empty words over the alphabet $\mathcal{V}$. By $\mathcal{V}^*$ we denote the set of all words over the alphabet $\mathcal{V}$, including the *empty word* ε.

Definition. Let α be a word over an alphabet $\mathcal{V}$. Then $|\alpha|$ (the *length* of α) is the number of symbols in α.

Definition. By $\mathcal{S}_{\mathrm{Free}}$ we denote the class of all free semigroups $\langle \mathcal{V}^+, \circ \rangle$, where $\mathcal{V}$ is a finite subset of a fixed countable alphabet $\{a_j \mid j \in \mathbf{N}\}$.

Definition. By $\mathcal{K}^m_{\mathrm{Free}}$ we denote the class of all $\mathrm{Tp}(m)$-quasimodels $\langle \mathcal{V}^+, \circ, v \rangle$, over free semigroups from $\mathcal{S}_{\mathrm{Free}}$, such that for every $A \in \mathrm{Tp}(m)$ there is $\alpha \in v(A)$ satisfying $|\alpha| \leq m$.

Lemma 5.1 *Let $m \in \mathbf{N}$, $\langle \mathcal{V}^+, \circ, v \rangle \in \mathcal{K}^m_{\mathrm{Free}}$, $\delta \in \mathcal{V}^*$, and $E \in \mathrm{Tp}(m)$.*
Then there is a $\mathrm{Tp}(m)$-quasimodel $\langle \mathcal{W}^+, \circ, w \rangle \in \mathcal{K}^m_{\mathrm{Free}}$ and there is $\alpha \in \mathcal{W}^+$ such that

(i) *$\langle \mathcal{W}^+, \circ, w \rangle$ is a conservative extension of $\langle \mathcal{V}^+, \circ, v \rangle$;*

(ii) *$\alpha \in w(E)$;*

(iii) *for any $F \in \mathrm{Tp}(m)$, if $L \not\vdash E \to F$, then $\alpha \notin w(F)$;*

(iv) *for any $F \in \mathrm{Tp}(m)$, if $\delta \in \mathcal{V}^+$ and $\delta \notin v(E \backslash F)$, then $\alpha \circ \delta \notin w(F)$;*

Proof. We are going to apply Lemma 4.1. First, we put $\mathbf{V} \rightleftharpoons \mathcal{V}^+$ and $n \rightleftharpoons |\delta| + 1$. Let $\mathbf{D}^{\mathrm{lf}}_{\mathrm{m,n}} = [0, k]$. Let x, z, y_1, y_2, … y_k be any $k+2$ distinct elements of $\{a_j \mid j \in \mathbf{N}\}$, which do not occur in $\mathcal{V}$. We denote $\mathcal{Y} \rightleftharpoons \{x, z, y_1, y_2, \ldots, y_k\}$ and put $\mathbf{W} \rightleftharpoons \mathcal{W}^+$, where $\mathcal{W} \rightleftharpoons \mathcal{V} \cup \mathcal{Y}$.

We define the function π as follows.

$$\pi\langle s, t \rangle \rightleftharpoons (x^m \circ y_{s+1} \circ z^m) \circ (x^m \circ y_{s+2} \circ z^m) \circ \ldots \circ (x^m \circ y_t \circ z^m)$$

Here $x^m \rightleftharpoons \underbrace{x \circ \ldots \circ x}_{m \text{ times}}$.

We define the function Subword: $\mathcal{W}^+ \to \mathbf{P}(\mathcal{W}^+)$ as

$$\mathrm{Subword}(\beta) \rightleftharpoons \{\alpha \in \mathcal{W}^+ \mid \beta = \gamma_1 \circ \alpha \circ \gamma_2 \;\text{ for some }\; \gamma_1, \gamma_2 \in \mathcal{W}^*\}.$$

Thus, Subword(β) is the set of all non-empty subwords of β. Next we introduce several subsets of $\mathcal{W}^+$.

$$\begin{aligned} \mathcal{R} &\rightleftharpoons \{\rho \in \mathcal{V}^+ \mid \rho \circ \alpha = \delta \;\text{ for some }\; \alpha \in \mathcal{V}^*\} \\ \mathcal{P}_0 &\rightleftharpoons \{\pi\langle s, t \rangle \mid 0 \leq s < t < k\} \\ \mathcal{P}_1 &\rightleftharpoons \{\pi\langle s, k \rangle \mid 0 \leq s < k\} \end{aligned}$$

$$
\begin{array}{rcl}
\mathcal{P}_2 & \rightleftharpoons & \mathcal{P}_1 \circ \mathcal{R} \\
\mathcal{P} & \rightleftharpoons & \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2 \\
\mathcal{M}_1 & \rightleftharpoons & \{\alpha \in \mathcal{W}^+ \mid \alpha \notin \mathcal{V}^+ \ \text{ and } \ \alpha \notin \text{Subword}(\pi\langle 0, k\rangle \circ \delta)\} \\
\mathcal{M}_2 & \rightleftharpoons & z \circ \mathcal{W}^* \\
\mathcal{M}_3 & \rightleftharpoons & \mathcal{W}^* \circ x \\
\mathcal{M} & \rightleftharpoons & \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \\
\mathcal{T} & \rightleftharpoons & \underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{m \text{ times}}
\end{array}
$$

Before applying Lemma 4.1 we establish several properties of these sets of words.

Lemma 5.1.1

(i) $\mathcal{W}^* \circ \mathcal{M}_1 \circ \mathcal{W}^* \subseteq \mathcal{M}_1$

(i') *If* $\beta \in \mathcal{W}^+$, $\alpha \in \text{Subword}(\beta)$, *and* $\alpha \in \mathcal{M}_1$, *then* $\beta \in \mathcal{M}_1$.

(ii) $\mathcal{M}_2 \circ \mathcal{W}^* \subseteq \mathcal{M}_2$

(iii) $\mathcal{W}^* \circ \mathcal{M}_3 \subseteq \mathcal{M}_3$

Proof. Obvious. ■

Lemma 5.1.2

(a) $\mathcal{P} \cap \mathcal{V}^+ = \emptyset$

(b) $\mathcal{P} \cap \mathcal{M} = \emptyset$

(c) $\mathcal{V}^+ \cap \mathcal{M} = \emptyset$

(d) $\mathcal{M} \circ \mathcal{M} \subseteq \mathcal{M}$

(e) $\mathcal{T} \subseteq \mathcal{M}$

(f) $\mathcal{M} \circ \mathcal{V}^+ \subseteq \mathcal{M}$

(g) $\mathcal{V}^+ \circ \mathcal{M} \subseteq \mathcal{M}$

(h) $\mathcal{P} \circ \mathcal{M} \subseteq \mathcal{T}$

(i) $\mathcal{M} \circ \mathcal{P} \subseteq \mathcal{T}$

(j) $\mathcal{T} \circ (\mathcal{P} \cup \mathcal{V}^+ \cup \mathcal{M}) \subseteq \mathcal{T}$

(k) $(\mathcal{P} \cup \mathcal{V}^+ \cup \mathcal{M}) \circ \mathcal{T} \subseteq \mathcal{T}$

(l) *If* $s \neq s'$ *then* $\pi\langle r, s\rangle \circ \pi\langle s', t\rangle \in \mathcal{T}$.

(m) $\mathcal{V}^+ \circ \mathcal{P} \subseteq \mathcal{T}$

(n) $\mathcal{P}_0 \circ \mathcal{V}^+ \subseteq \mathcal{T}$

(o) $\mathcal{P}_1 \circ \{\beta \in \mathcal{W}^+ \mid \beta \notin \mathcal{R}\} \subseteq \mathcal{T}$

(p) $\mathcal{P} \circ \mathcal{V}^+ \subseteq \mathcal{P} \cup \mathcal{T}$

(q) $\mathcal{P} \circ \mathcal{P} \subseteq \mathcal{P} \cup \mathcal{T}$

PROOF.

(a)

Evident from $\mathcal{P} \subseteq \mathcal{Y}^+ \circ \mathcal{V}^*$.

(b)

Let $\alpha \in \mathcal{P}$. Then the leftmost symbol of α is x and the rightmost symbol of α belongs to $\mathcal{V} \cup \{z\}$. Thus $\alpha \notin \mathcal{M}_2$ and $\alpha \notin \mathcal{M}_3$. Note that $\mathcal{P} \subseteq \text{Subword}(\pi\langle 0, k\rangle \circ \delta)$. Thus $\alpha \notin \mathcal{M}_1$.

(c)

Obvious.

(d)

Let $\alpha \in \mathcal{M}$ and $\beta \in \mathcal{M}$. We verify that $\alpha \circ \beta \in \mathcal{M}$. If $\alpha \in \mathcal{M}_1$ then $\alpha \circ \beta \in \mathcal{M}_1$. If $\alpha \in \mathcal{M}_2$ then $\alpha \circ \beta \in \mathcal{M}_2$. If $\beta \in \mathcal{M}_1$ then $\alpha \circ \beta \in \mathcal{M}_1$. If $\beta \in \mathcal{M}_3$ then $\alpha \circ \beta \in \mathcal{M}_3$. The only complicated case is $\alpha \in \mathcal{M}_3$ and $\beta \in \mathcal{M}_2$, i.e., $\alpha = \alpha' \circ x$ and $\beta = z \circ \beta'$. Note that then $x \circ z \in \text{Subword}(\alpha \circ \beta)$ and $x \circ z \in \mathcal{M}_1$. It remains to apply Lemma 5.1.1 (i').

(e)

Follows from (d).

(f)

Let $\alpha \in \mathcal{M}$ and $\beta \in \mathcal{V}^+$. We verify that $\alpha \circ \beta \in \mathcal{M}$. If $\alpha \in \mathcal{M}_1$ then $\alpha \circ \beta \in \mathcal{M}_1$. If $\alpha \in \mathcal{M}_2$ then $\alpha \circ \beta \in \mathcal{M}_2$. The only complicated case is $\alpha = \alpha' \circ x$. In this case $x \circ \beta \in \text{Subword}(\alpha \circ \beta)$ and $x \circ \beta \in \mathcal{M}_1$.

(g)

Let $\alpha \in \mathcal{V}^+$ and $\beta \in \mathcal{M}$. We verify that $\alpha \circ \beta \in \mathcal{M}$. If $\beta \in \mathcal{M}_1$ then $\alpha \circ \beta \in \mathcal{M}_1$. If $\beta \in \mathcal{M}_3$ then $\alpha \circ \beta \in \mathcal{M}_3$. The only complicated case is $\beta = z \circ \beta'$. Note that in that case $\alpha \circ z \in \text{Subword}(\alpha \circ \beta)$ and $\alpha \circ z \in \mathcal{M}_1$.

In the following part of the proof we denote by $\pi\langle s, s\rangle$ the empty word in $\mathcal{W}^*$.

(h)

Let $\gamma \in \mathcal{P} \circ \mathcal{M}$.

CASE 1: $\gamma = \pi\langle s, t\rangle \circ \beta$, $\beta \in \mathcal{M}$

Evidently $\gamma = \underbrace{x \circ \ldots \circ x}_{m-1 \text{ times}} \circ \phi$, where $\phi = x \circ y_{s+1} \circ z^m \circ \pi\langle s+1, t\rangle \circ \beta$. We must verify that $\phi \in \mathcal{M}$.

If $\beta \in \mathcal{M}_1$ then $\phi \in \mathcal{M}_1$. If $\beta \in \mathcal{M}_3$ then $\phi \in \mathcal{M}_3$. Let now $\beta \in \mathcal{M}_2$, i.e. $\beta = z \circ \beta'$. Evidently $z^{m+1} \in \text{Subword}(y_{s+1} \circ z^m \circ \pi\langle s+1, t\rangle \circ z \circ \beta'$ and $z^{m+1} \in \mathcal{M}_1$.

CASE 2: $\gamma = \pi\langle s,k\rangle \circ \rho \circ \beta$, $\beta \in \mathcal{M}$, $\rho \in \mathcal{R}$
Evidently $\gamma = \underbrace{x \circ \ldots \circ x}_{m-1 \text{ times}} \circ \phi$, where $\phi = x \circ y_{s+1} \circ z^m \circ \pi\langle s+1,k\rangle \circ \rho \circ \beta$. The only complicated case is $\beta \in \mathcal{M}_2$, i.e., $\beta = z \circ \beta'$. Note that $\rho \circ z \in \text{Subword}(\phi)$ and $\rho \circ z \in \mathcal{M}_1$.

(i) and (m)
Let $\alpha \in \mathcal{M} \cup \mathcal{V}^+$ and $\beta \in \mathcal{P}$. We must prove that $\alpha \circ \beta \in \underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{m \text{ times}}$.
CASE 1: $\beta = \pi\langle s,t\rangle$
Evidently $\alpha \circ \beta = \phi \circ \underbrace{z \circ \ldots \circ z}_{m-1 \text{ times}}$, where $\phi = (\alpha \circ \pi\langle s,t-1\rangle \circ x^m \circ y_t \circ z)$. Obviously $z \in \mathcal{M}_2$.
It remains to verify that $\phi \in \mathcal{M}$.
CASE 1a: $\alpha \in \mathcal{M}_1$
Obvious from Lemma 5.1.1 (i).
CASE 1b: $\alpha \in \mathcal{M}_2$
Obvious from Lemma 5.1.1 (ii).
CASE 1c: $\alpha \in \mathcal{M}_3$
Note that the rightmost symbol of α is x and the first m symbols of $\pi\langle s,t-1\rangle \circ x^m \circ y_t \circ z$ are x^m. Thus $x^{m+1} \in \text{Subword}(\phi)$. In view of $x^{m+1} \in \mathcal{M}_1$ we have $\phi \in \mathcal{M}_1$.
CASE 1d: $\alpha \in \mathcal{V}^+$
Evidently $\alpha \circ x \in \text{Subword}(\phi)$. On the other hand, $\alpha \circ x \in \mathcal{V}^+ \circ \mathcal{Y}^+$ and $\mathcal{V}^+ \circ \mathcal{Y}^+ \subseteq \mathcal{M}_1$. According to Lemma 5.1.1 (i'), $\phi \in \mathcal{M}_1$.
CASE 2: $\beta = \pi\langle s,k\rangle \circ \rho$, $\rho \in \mathcal{R}$
Now $\alpha \circ \beta = \phi \circ \underbrace{z \circ \ldots \circ z}_{m-2 \text{ times}} \circ (z \circ \rho)$, where ϕ is the same as in the previous case. We have already verified that $z \in \mathcal{M}$ and $\phi \in \mathcal{M}$. Evidently also $z \circ \rho \in \mathcal{M}$.

(j)
In view of (i) and (e) we have $\mathcal{T} \circ \mathcal{P} = \mathcal{M}^m \circ \mathcal{P} \subseteq \mathcal{M}^{m-1} \circ \mathcal{T} \subseteq \mathcal{M}^m = \mathcal{T}$. From (f) we obtain $\mathcal{T} \circ \mathcal{V}^+ = \mathcal{M}^m \circ \mathcal{V}^+ \subseteq \mathcal{M}^m = \mathcal{T}$. According to (d) we have $\mathcal{T} \circ \mathcal{M} = \mathcal{M}^{m+1} \subseteq \mathcal{M}^m = \mathcal{T}$.

(k)
Similar. We use (h), (e), (g), and (d).

(l)
Evidently $\pi\langle r,s\rangle \circ \pi\langle s',t\rangle = \phi \circ \underbrace{z \circ \ldots \circ z}_{m-1 \text{ times}}$, where $\phi = (\pi\langle r,s\rangle \circ \pi\langle s',t-1\rangle \circ x^m \circ y_t \circ z)$. We only need to prove that $\phi \in \mathcal{M}$. Note that $y_s \circ z^m \circ x^m \circ y_{s'+1} \in \phi$. On the other hand $y_s \circ z^m \circ x^m \circ y_{s'+1} \in \mathcal{M}_1$, since $s \neq s'$. According to Lemma 5.1.1 (i'), $\phi \in \mathcal{M}_1$.

(n)
Let $\alpha = \pi\langle s,t\rangle$, $t < k$, and $\beta \in \mathcal{V}^+$. Evidently $\alpha \circ \beta = \underbrace{x \circ \ldots \circ x}_{m-1 \text{ times}} \circ \phi$, where $\phi = x \circ y_{s+1} \circ z^m \circ \pi\langle s+1,t\rangle \circ \beta$. Note that $y_t \circ z^m \circ \beta \in \text{Subword}(\phi)$. On the other hand, $y_t \circ z^m \circ \beta \in \mathcal{M}_1$, since $t \neq k$. Thus $\phi \in \mathcal{M}_1$.

(o)
Let $\alpha = \pi\langle s,k\rangle$ and $\beta \notin \mathcal{R}$. Evidently $\alpha \circ \beta = \underbrace{x \circ \ldots \circ x}_{m-1 \text{ times}} \circ \phi$, where

$\phi = x \circ y_{s+1} \circ z^m \circ \pi\langle s+1, k\rangle \circ \beta$. Note that $z \circ \beta \in \mathrm{Subword}(\phi)$. On the other hand, $z \circ \beta \in \mathcal{M}_1$, since β is not a left subword of δ (see the definition of $\mathcal{R}$). Thus $\phi \in \mathcal{M}_1$.

(p)

Let $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{V}^+$. We must prove that $\alpha \circ \beta \in \mathcal{P} \cup \mathcal{T}$.

CASE 1: $\alpha \in \mathcal{P}_0$

According to (n), $\alpha \circ \beta \in \mathcal{T}$.

CASE 2: $\alpha \in \mathcal{P}_1$

If $\beta \in \mathcal{R}$ then $\alpha \circ \beta \in \mathcal{P}_2$. If $\beta \notin \mathcal{R}$, then $\alpha \circ \beta \in \mathcal{T}$ in view of (o).

CASE 3: $\alpha \in \mathcal{P}_2$

Evidently $\mathcal{P}_2 \circ \mathcal{V}^+ = (\mathcal{P}_1 \circ \mathcal{R}) \circ \mathcal{V}^+ = \mathcal{P}_1 \circ (\mathcal{R} \circ \mathcal{V}^+) \subseteq \mathcal{P}_1 \circ \mathcal{V}^+$ and we can apply case 2.

(q)

Let $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{P}$. We must prove that $\alpha \circ \beta \in \mathcal{P} \cup \mathcal{T}$.

CASE 1: $\alpha \in \mathcal{P}_0 \cup \mathcal{P}_1$, i.e., $\alpha = \pi\langle r, s\rangle$, where $0 \le r < s \le k$

CASE 1a: $\beta \in \mathcal{P}_0 \cup \mathcal{P}_1$, i.e., $\beta = \pi\langle s', t\rangle$, where $0 \le s', t \le k$

If $s = s'$, then $\alpha \circ \beta = \pi\langle r, s\rangle \circ \pi\langle s, t\rangle = \pi\langle r, t\rangle \in \mathcal{P}$ according to the definition of the function π. If $s \neq s'$, then $\alpha \circ \beta \in \mathcal{T}$ according to (l).

CASE 1b: $\beta \in \mathcal{P}_2$

Evidently $\alpha \circ \beta \in \alpha \circ \mathcal{P}_1 \circ \mathcal{R}$. According to case 1a, $\alpha \circ \beta \in (\mathcal{P} \cup \mathcal{T}) \circ \mathcal{R} \subseteq (\mathcal{P} \cup \mathcal{T}) \circ \mathcal{V}^+$. From (p) and (j) we obtain $(\mathcal{P} \cup \mathcal{T}) \circ \mathcal{V}^+ \subseteq \mathcal{P} \cup \mathcal{T}$.

CASE 2: $\alpha \in \mathcal{P}_2$

From (m) and (k) we get $\mathcal{P}_2 \circ \mathcal{P} = \mathcal{P}_1 \circ \mathcal{R} \circ \mathcal{P} \subseteq \mathcal{P}_1 \circ (\mathcal{V}^+ \circ \mathcal{P}) \subseteq \mathcal{P}_1 \circ \mathcal{T} \subseteq \mathcal{T}$.

This completes the proof of Lemma 5.1.2. ■

Now we apply Lemma 4.1 and obtain a function $u\colon \mathrm{Tp}(m) \to \mathbf{P}(\mathbf{W})$ satisfying the conditions (i)–(vii) from Lemma 4.1.

We define a function $\mathrm{Subst}_{\mathcal{M}}\colon \mathcal{W}^+ \to \mathbf{P}(\mathcal{W}^+)$ and two valuations $w_0\colon \mathrm{Tp}(m) \to \mathbf{P}(\mathcal{W}^+)$ and $w\colon \mathrm{Tp}(m) \to \mathbf{P}(\mathcal{W}^+)$.

$$
\begin{aligned}
\mathrm{Subst}_{\mathcal{M}}(q) &\rightleftharpoons \mathcal{M} \cup \{q\} \text{ if } q \in \mathcal{W} \\
\mathrm{Subst}_{\mathcal{M}}(\alpha \circ q) &\rightleftharpoons \mathrm{Subst}_{\mathcal{M}}(\alpha) \circ (\{q\} \cup \mathcal{M}) \text{ if } \alpha \in \mathcal{W}^+ \text{ and } q \in \mathcal{W}
\end{aligned}
$$

The set $\mathrm{Subst}_{\mathcal{M}}(\alpha)$ consists of all words that are obtained replacing some (may be none) of symbol occurrences in α by words from the set $\mathcal{M}$.

$$
\begin{aligned}
w_0(A) &\rightleftharpoons \bigcup_{\alpha \in v(A)} \mathrm{Subst}_{\mathcal{M}}(\alpha, \mathcal{M}) \\
w(A) &\rightleftharpoons u(A) \cup w_0(A)
\end{aligned}
$$

Lemma 5.1.3 *Let* $\alpha \in \mathcal{W}^+$ *and* $\beta \in \mathcal{W}^+$. *Then* $\mathrm{Subst}_{\mathcal{M}}(\alpha \circ \beta) = \mathrm{Subst}_{\mathcal{M}}(\alpha) \circ \mathrm{Subst}_{\mathcal{M}}(\beta)$.

PROOF. Induction on $|\beta|$. ■

Lemma 5.1.4 *Let $A \in \mathrm{Tp}(m)$. Then*

(i) $v(A) \subseteq w_0(A)$;

(ii) $w_0(A) \subseteq v(A) \cup \mathcal{M}$.

PROOF. It suffices to verify that, for any $\alpha \in v(A)$,

(i) $\alpha \in \mathrm{Subst}_{\mathcal{M}}(\alpha)$;

(ii) $\mathrm{Subst}_{\mathcal{M}}(\alpha) \subseteq \{\alpha\} \cup \mathcal{M}$.

We prove this by induction on $|\alpha|$ for any $\alpha \in \mathcal{V}^+$.

Induction step.

(i)

Let $\alpha \in \mathcal{V}^+$, $q \in \mathcal{V}$, and $\alpha \in \mathrm{Subst}_{\mathcal{M}}(\alpha)$. Then $\alpha \circ q \in \mathrm{Subst}_{\mathcal{M}}(\alpha) \circ \{q\} \subseteq \mathrm{Subst}_{\mathcal{M}}(\alpha \circ q)$.

(ii)

Let $\alpha \in \mathcal{V}^+$, $q \in \mathcal{V}$, and $\mathrm{Subst}_{\mathcal{M}}(\alpha) \subseteq \{\alpha\} \cup \mathcal{M}$. Then $\mathrm{Subst}_{\mathcal{M}}(\alpha \circ q) = \mathrm{Subst}_{\mathcal{M}}(\alpha) \circ (\{q\} \cup \mathcal{M}) \subseteq (\{\alpha\} \cup \mathcal{M}) \circ (\{q\} \cup \mathcal{M}) = \{\alpha \circ q\} \cup (\{\alpha\} \circ \mathcal{M}) \cup (\mathcal{M} \circ \{q\}) \cup (\mathcal{M} \circ \mathcal{M})$. From Lemma 5.1.2 (g), (f), and (d) we obtain $\mathrm{Subst}_{\mathcal{M}}(\alpha \circ q) \subseteq \{\alpha \circ q\} \cup \mathcal{M}$. ■

Lemma 5.1.5 *Let $A \in \mathrm{Tp}(m)$. Then $\mathcal{T} \subseteq w(A)$.*

PROOF. Since $\langle \mathcal{V}^+, \circ, v \rangle \in \mathcal{K}^m_{\mathrm{Free}}$, we can choose a word $\alpha \in v(A)$ such that $|\alpha| \leq m$. Evidently $\underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{|\alpha| \text{ times}} \subseteq \mathrm{Subst}_{\mathcal{M}}(\alpha) \subseteq w_0(A) \subseteq w(A)$. In view of $\mathcal{M} \circ \mathcal{M} \subseteq \mathcal{M}$ and taking into account that $|\alpha| \leq m$, we have $\underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{m \text{ times}} \subseteq \underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{|\alpha| \text{ times}}$. Thus $\mathcal{T} = \underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{m \text{ times}} \subseteq w(A)$. ■

In order to make the formulation of the next lemma more readable we introduce the following two subsets of $\mathcal{W}^*$ (recall that ε stands for the empty word).

$$\begin{aligned} \mathcal{Q} &\rightleftharpoons \mathcal{P} \cup \mathcal{V}^+ \cup \mathcal{M} \\ \mathcal{Q}_\infty &\rightleftharpoons \{\varepsilon\} \cup \mathcal{Q} \cup (\mathcal{Q} \circ \mathcal{Q}) \cup (\mathcal{Q} \circ \mathcal{Q} \circ \mathcal{Q}) \cup \ldots \end{aligned}$$

Lemma 5.1.6

(i) $\mathcal{Q}_\infty \circ \mathcal{P} \circ \mathcal{Q}_\infty \subseteq \mathcal{P} \cup \mathcal{T}$

(ii) $\mathcal{Q}_\infty \circ \mathcal{P} \circ \mathcal{Q}_\infty \circ \mathcal{M} \circ \mathcal{Q}_\infty \subseteq \mathcal{T}$

(iii) $\mathcal{Q}_\infty \circ \mathcal{M} \circ \mathcal{Q}_\infty \circ \mathcal{P} \circ \mathcal{Q}_\infty \subseteq \mathcal{T}$

Proof.

(i)

From Lemma 5.1.2 (q), (m), (i) and (k) we obtain $\mathcal{Q} \circ (\mathcal{P} \cup \mathcal{T}) \subseteq \mathcal{P} \cup \mathcal{T}$. Now we can easily prove $\underbrace{\mathcal{Q} \circ \ldots \circ \mathcal{Q}}_{l \text{ times}} \circ \mathcal{P} \subseteq \mathcal{P} \cup \mathcal{T}$ by induction on l.

From Lemma 5.1.2 (q), (p), (h) and (j) we obtain $(\mathcal{P} \cup \mathcal{T}) \circ \mathcal{Q} \subseteq \mathcal{P} \cup \mathcal{T}$. By induction on l we see that $\mathcal{Q}_\infty \circ \mathcal{P} \circ \underbrace{\mathcal{Q} \circ \ldots \circ \mathcal{Q}}_{l \text{ times}} \subseteq \mathcal{P} \cup \mathcal{T}$.

(ii)

We prove that $(\mathcal{Q}_\infty \circ \mathcal{P} \circ \mathcal{Q}_\infty) \circ \mathcal{M} \circ \underbrace{\mathcal{Q} \circ \ldots \circ \mathcal{Q}}_{l \text{ times}} \subseteq \mathcal{T}$ by induction on l.

Induction base. First we apply (i). Further, from Lemma 5.1.2 (h) and (j) we obtain $(\mathcal{P} \cup \mathcal{T}) \circ \mathcal{M} \subseteq \mathcal{T}$.

Induction step. From Lemma 5.1.2 (j) we see that $\mathcal{T} \circ \mathcal{Q} \subseteq \mathcal{T}$.

(iii)

We prove that $\underbrace{\mathcal{Q} \circ \ldots \circ \mathcal{Q}}_{l \text{ times}} \circ \mathcal{M} \circ (\mathcal{Q}_\infty \circ \mathcal{P} \circ \mathcal{Q}_\infty) \subseteq \mathcal{T}$ by induction on l.

Induction base. From Lemma 5.1.2 (i) and (k) we obtain $\mathcal{M} \circ (\mathcal{P} \cup \mathcal{T}) \subseteq \mathcal{T}$.

Induction step. From Lemma 5.1.2 (k) we see that $\mathcal{Q} \circ \mathcal{T} \subseteq \mathcal{T}$. ■

Lemma 5.1.7 $\langle \mathcal{W}^+, \circ, w \rangle$ *is a* Tp(m)*-quasimodel.*

Proof. We verify the conditions (1') and (2) from the definition of a Tp(m)-quasimodel at page 5.

(1')

Let $A{\bullet}B \in \mathrm{Tp}(m)$ and $\gamma \in w(A{\bullet}B)$. We must prove that $\gamma \in w(A) \circ w(B)$.

Case 1: $\gamma \in u(A{\bullet}B)$

Obvious from Lemma 4.1 (i).

Case 2: $\gamma \in w_0(A{\bullet}B)$

Evidently $\gamma \in \mathrm{Subst}_{\mathcal{M}}(\gamma')$ for some $\gamma' \in v(A{\bullet}B) = v(A) \circ v(B)$. Thus $\gamma' = \alpha' \circ \beta'$, where $\alpha' \in v(A)$ and $\beta' \in v(B)$. According to Lemma 5.1.3,
$\mathrm{Subst}_{\mathcal{M}}(\gamma') = \mathrm{Subst}_{\mathcal{M}}(\alpha') \circ \mathrm{Subst}_{\mathcal{M}}(\beta') \subseteq w_0(A) \circ w_0(B)$.

(2)

Let $A_1, \ldots, A_l, B \in \mathrm{Tp}(m)$, $L \vdash A_1 \ldots A_l \to B$, $\alpha_1 \in w(A_1), \ldots \alpha_l \in w(A_l)$. We must prove that $\alpha_1 \circ \ldots \circ \alpha_l \in w(B)$.

Case 1: $(\forall i \le l)\, \alpha_i \in u(A_i)$

According to Lemma 4.1 (ii), $\alpha_1 \circ \ldots \circ \alpha_l \in u(B) \cup \mathcal{T}$.

In view of Lemma 5.1.5, $\alpha_1 \circ \ldots \circ \alpha_l \in w(B)$.

Case 2: $(\forall j \le l)\, \alpha_j \in w_0(A_j)$

This means that for every number $j \le l$ there is a word $\beta_j \in v(A_j)$ such that $\alpha_j \in \mathrm{Subst}_{\mathcal{M}}(\beta_j)$. According to Lemma 5.1.3, $\alpha_1 \circ \ldots \circ \alpha_l \in \mathrm{Subst}_{\mathcal{M}}(\beta_1 \circ \ldots \circ \beta_l)$. Note that $\beta_1 \circ \ldots \circ \beta_l \in v(A_1) \circ \ldots \circ v(A_l) \subseteq v(B)$, since $\langle \mathcal{V}^+, \circ, v \rangle$ is a Tp(m)-quasimodel. Thus $\alpha_1 \circ \ldots \circ \alpha_l \in w_0(B)$.

CASE 3: $(\exists i \leq l)\, \alpha_i \notin u(A_i)$ and $(\exists j \leq l)\, \alpha_j \notin w_0(A_j)$
Evidently $\alpha_i \in w_0(A_i)$. From Lemma 5.1.4 (ii) and Lemma 4.1 (vi) we obtain $\alpha_i \in v(A_i) \cup \mathcal{M}$ and $\alpha_i \notin v(A_i)$ respectively. Thus $\alpha_i \in \mathcal{M}$.

Evidently $\alpha_j \in u(A_j)$. From Lemma 4.1 (vii) and Lemma 5.1.4 (i) we obtain $\alpha_j \in v(A_j) \cup \mathcal{P}$ and $\alpha_j \notin v(A_j)$ respectively. Thus $\alpha_j \in \mathcal{P}$.

Note that $\alpha_{k'} \in \mathcal{Q}$ for every $k' \leq l$. According to Lemma 5.1.6 (i) and (ii), $\alpha_1 \circ \ldots \circ \alpha_l \in \mathcal{T}$. It remains to apply Lemma 5.1.5. ■

We continue the proof of Lemma 5.1. The desired word $\alpha \in \mathcal{W}^+$ is taken to be $\alpha \rightleftharpoons \pi\langle g, k\rangle$.

(i)
Let $A \in \mathrm{Tp}(m)$. We must verify that $w(A) \cap \mathcal{V}^+ = v(A)$.

From the definition of w we see that $w(A) \cap \mathcal{V}^+ = (u(A) \cap \mathcal{V}^+) \cup (w_0(A) \cap \mathcal{V}^+)$. According to Lemma 4.1 (vi) and (vii), and Lemma 5.1.2 (a) we have $u(A) \cap \mathcal{V}^+ = v(A)$. On the other hand, in view of Lemma 5.1.4 and Lemma 5.1.2 (c), $w_0(A) \cap \mathcal{V}^+ = v(A)$.

(ii)
Immediate from Lemma 4.1 (iii).

(iii)
Immediate from Lemma 4.1 (iv), if we take into account that $\pi\langle g, k\rangle \notin \mathcal{V}^+ \cup \mathcal{M}$ and thus $\pi\langle g, k\rangle \notin w_0(F)$ for any $F \in \mathrm{Tp}(m)$.

(iv)
Immediate from Lemma 4.1 (v), if we take into account that $\pi\langle g, k\rangle \circ \delta \notin \mathcal{V}^+ \cup \mathcal{M}$ and thus $\pi\langle g, k\rangle \circ \delta \notin w_0(F)$ for any $F \in \mathrm{Tp}(m)$. ■

Before proving that the Lambek calculus is L-complete we have to verify that the class $\mathcal{K}^m_{\mathrm{Free}}$ is not empty.

Lemma 5.2 *The class $\mathcal{K}^m_{\mathrm{Free}}$ is not empty.*

PROOF.
We define the *positive count* $\bar{\#}$ as the following mapping from types to positive integers.

$$
\begin{aligned}
\bar{\#}p_i &\rightleftharpoons 1 \\
\bar{\#}(A\bullet B) &\rightleftharpoons \bar{\#}A + \bar{\#}B \\
\bar{\#}(A\backslash B) &\rightleftharpoons \max(1, \bar{\#}B - \bar{\#}A) \\
\bar{\#}(A/B) &\rightleftharpoons \max(1, \bar{\#}A - \bar{\#}B)
\end{aligned}
$$

The positive count of a sequence of types is defined in the natural way.

$$\bar{\#}(A_1 \ldots A_l) \rightleftharpoons \bar{\#}A_1 + \ldots + \bar{\#}A_l$$

Lemma 5.2.1 *For any type A, $\bar{\#}A \leq \|A\|$.*

Lemma 5.2.2 *If $L \vdash \Gamma \to A$ then $\bar{\#}\Gamma \geq \bar{\#}A$.*

Proof. Straightforward induction on the length of the derivation.
Case 1: Axiom
Obvious.
Case 2: $(\to\backslash)$ Given $\dfrac{A\ \Pi\to B}{\Pi\to A\backslash B}\,(\to\backslash)$ where $\Pi \neq \Lambda$.
By the induction hypothesis $\bar{\#}A + \bar{\#}\Pi \geq \bar{\#}B$, whence $\bar{\#}\Pi \geq \bar{\#}B - \bar{\#}A$. On the other hand, for any non-empty sequence of types Π, $\bar{\#}\Pi \geq 1$.
Thus $\bar{\#}\Pi \geq \max(1, \bar{\#}B - \bar{\#}A) = \bar{\#}(A\backslash B)$.
Case 3: $(\to/)$
Similar.
Case 4: $(\backslash\to)$ Given $\dfrac{\Phi\to A \quad \Gamma B\Delta\to C}{\Gamma\Phi(A\backslash B)\Delta\to C}\,(\backslash\to)$
By the induction hypothesis $\bar{\#}\Phi \geq \bar{\#}A$ and $\bar{\#}\Gamma + \bar{\#}B + \bar{\#}\Delta \geq \bar{\#}C$.
Note that $\bar{\#}(A\backslash B) \geq \bar{\#}B - \bar{\#}A$.
Hence $\bar{\#}\Gamma + \bar{\#}\Phi + \bar{\#}(A\backslash B) + \bar{\#}\Delta \geq \bar{\#}\Gamma + \bar{\#}A + (\bar{\#}B - \bar{\#}A) + \bar{\#}\Delta \geq \bar{\#}C$.
Case 5: $(/\to)$
Similar.
Case 6: $(\to\bullet)$ Given $\dfrac{\Gamma\to A \quad \Delta\to B}{\Gamma\Delta\to A\bullet B}\,(\to\bullet)$
If $\bar{\#}\Gamma \geq \bar{\#}A$ and $\bar{\#}\Delta \geq \bar{\#}B$, then $\bar{\#}\Gamma + \bar{\#}\Delta \geq \bar{\#}A + \bar{\#}B = \bar{\#}(A\bullet B)$.
Case 7: $(\bullet\to)$ Given $\dfrac{\Gamma AB\Delta\to C}{\Gamma(A\bullet B)\Delta\to C}\,(\bullet\to)$
Evidently $\bar{\#}(\Gamma(A\bullet B)\Delta) = \bar{\#}(\Gamma AB\Delta)$. ■

Now we define a $\mathrm{Tp}(m)$-quasimodel $\langle \mathcal{V}^+, \circ, v\rangle$.

$$\mathcal{V} \rightleftharpoons \{a_0\} \qquad v(A) \rightleftharpoons \{\underbrace{a_0 \circ \ldots \circ a_0}_{k\ \text{times}} \mid k \geq \bar{\#}A\}$$

It is immediate from Lemma 5.2.1 and Lemma 5.2.2 that $\langle \mathcal{V}^+, \circ, v\rangle \in \mathcal{K}^m_{\mathrm{Free}}$. This completes the proof of Lemma 5.2. ■

Theorem 3 *The Lambek calculus is complete with respect to the class of all language models (L-models).*

Proof. Let $L \nvdash E\to F$. We are going to prove that there is an L-model $\langle \mathcal{W}^+, \circ, w\rangle$ such that

(i) $\mathcal{W} \subseteq \{a_j \mid j \in \mathbf{N}\}$;

(ii) $w(E) \not\subseteq w(F)$.

Evidently, there is a natural number m such that $E \in \mathrm{Tp}(m)$ and $F \in \mathrm{Tp}(m)$. We apply Lemma 5.1 putting $\delta = \varepsilon$ (the empty word) and taking any L-model from $\mathcal{K}^m_{\mathrm{Free}}$ as $\langle \mathcal{V}^+, \circ, v\rangle$ (cf. Lemma 5.2). Thus we obtain an L-model $\langle \mathcal{W}_0^+, \circ, w_0\rangle \in \mathcal{K}^m_{\mathrm{Free}}$ such that $w_0(E) \not\subseteq w_0(F)$ (see Lemma 5.1 (ii) and (iii)).

To apply Theorem 1, we must first verify that the class $\mathcal{K}^m_{\mathrm{Free}}$ is witnessed. This follows from Lemma 5.1 (i), (ii), and (iv). ■

Theorem 4 *The Lambek calculus is complete with respect to the class of all language models over a two symbol alphabet* $\{b, c\}$.

PROOF. Let $L \not\vdash E \to F$. Following the proof of Theorem 3 we find a free semigroup model $\langle \mathcal{V}^+, \circ, v\rangle$, where $\mathcal{V} \subseteq \{a_j \mid j \in \mathbf{N}\}$, such that $v(E) \not\subseteq v(F)$ and $v(A) \neq \emptyset$ for every $A \in \mathrm{Tp}(m)$. We take $\mathcal{W} \rightleftharpoons \{b, c\}$ and define a function $g\colon \mathcal{V}^+ \to \mathcal{W}^+$ as follows.

$$g(a_j) \rightleftharpoons b \circ \underbrace{c \circ \ldots \circ c}_{j \text{ times}} \circ b \qquad g(\alpha \circ \beta) \rightleftharpoons g(\alpha) \circ g(\beta)$$

Now we put $w(p_i) \rightleftharpoons \{g(\gamma) \mid \gamma \in v(p_i)\}$ for every primitive type p_i and define $w(A)$ for complex types like in the proof of Theorem 1.

By induction on $\|A\|$ we see that $w(A) = \{g(\gamma) \mid \gamma \in v(A)\}$ for every $A \in \mathrm{Tp}(m)$. Thus $w(E) \not\subseteq w(F)$. ■

6 Weights

In this section we assign to every derivable sequent of the form $\Gamma \to B \bullet C$ a set of positive integers. These integers are "weights" of the type B with respect to different derivations of $\Gamma \to B \bullet C$. The main properties of the weights are the following.

- Given a fixed sequence Γ, there is only a finite number of possible values for the weights of B with respect to derivations of $\Gamma \to B \bullet C$.
- If the weights of B_1 and B_2 with respect to some derivations of $\Gamma \to B_1 \bullet C_1$ and $\Gamma \to B_2 \bullet C_2$ are equal, then the sequents $\Gamma \to B_1 \bullet C_2$ and $\Gamma \to B_2 \bullet C_1$ are derivable in the Lambek calculus (cf. Lemma 6.8).

At the end of this section we shall prove Lemma 3.1. The domain $\mathbf{D}_\Gamma$ of the quasimodel $\langle \mathbf{V}_\Gamma \subset \mathbf{D}_\Gamma \times \mathbf{D}_\Gamma, \circ, v_\Gamma \rangle$ will consist of all possible values for the weights corresponding to the antecedent Γ.

6.1 Calculus L^μ with multiple succedents

Here we introduce an alternative axiomatization of the Lambek calculus.

The sequents of L^μ are of the form $\Gamma \to \Delta$, where Γ and Δ are non-empty sequences of types. The intended interpretation of $A_1 \ldots A_m \to B_1 \ldots B_n$ is $A_1 \bullet \ldots \bullet A_m \to B_1 \bullet \ldots \bullet B_n$.

The axiom scheme is $\Pi \to \Pi$, where $\Pi \neq \Lambda$.

The rules of L^μ are the following.

$$\frac{A\Pi \to B}{\Pi \to A \backslash B}\ (\to\backslash) \quad \text{where} \quad \Pi \neq \Lambda \qquad\qquad \frac{\Phi \to A \quad \Gamma B \Delta \to \Theta}{\Gamma \Phi (A \backslash B) \Delta \to \Theta}\ (\backslash\to)$$

$$\frac{\Pi A \to B}{\Pi \to B / A}\ (\to/) \quad \text{where} \quad \Pi \neq \Lambda \qquad\qquad \frac{\Phi \to A \quad \Gamma B \Delta \to \Theta}{\Gamma (B / A) \Phi \Delta \to \Theta}\ (/\to)$$

$$\frac{\Gamma\to\Theta AB\,\Xi}{\Gamma\to\Theta(A\bullet B)\Xi}\ (\to\bullet) \qquad \frac{\Gamma AB\Delta\to\Theta}{\Gamma(A\bullet B)\Delta\to\Theta}\ (\bullet\to)$$

$$\frac{\Gamma\to\Theta \quad \Delta\to\Xi}{\Gamma\Delta\to\Theta\Xi}\ (CON)$$

We shall label L^μ-derivations with symbols $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}'$, etc. We write $L^\mu\vdash\Gamma\overset{\mathcal{D}}{\to}\Theta$ for '$\mathcal{D}$ is an L^μ-derivation of $\Gamma\to\Theta$'.

6.2 Equivalence of L^μ and L

Lemma 6.1 *$L^\mu\vdash\Gamma\to A_1\dots A_n$ if and only if $L\vdash\Gamma\to A_1\bullet\dots\bullet A_n$.*

Proof. 'If' part.
Straightforward induction on the length of a cutfree derivation of $\Gamma\to A_1\bullet\dots\bullet A_n$.

'Only if' part.
Induction on the length of the L^μ-derivation of $\Gamma\to A_1\dots A_n$.

In the case of the rule

$$\frac{\Gamma\to C_1\bullet\dots\bullet C_m ABD_1\bullet\dots\bullet D_n}{\Gamma\to C_1\bullet\dots\bullet C_m(A\bullet B)D_1\bullet\dots\bullet D_n}\ (\to\bullet)$$

we apply cut with the sequent
$((C_1\bullet\dots\bullet C_m\bullet A)\bullet B)\bullet D_1\bullet\dots\bullet D_n\to(C_1\bullet\dots\bullet C_m\bullet(A\bullet B))\bullet D_1\bullet\dots\bullet D_n$.

In the case of the rule

$$\frac{\Gamma\to A_1\dots A_n \quad \Delta\to B_1\dots B_m}{\Gamma\Delta\to A_1\dots A_nB_1\dots B_m}\ (CON)$$

we apply cut with the sequent $C\bullet(\dots(B_1\bullet B_2)\dots\bullet B_m)\to(\dots((C\bullet B_1)\bullet B_2)\dots\bullet B_m)$, where $C=A_1\bullet\dots\bullet A_n$.

Other cases are trivial. ■

Lemma 6.2 *The rule*

$$\frac{\Gamma\to\Delta \quad \Delta\to\Pi}{\Gamma\to\Pi}\ (CUT)$$

is admissible in the calculus L^μ.

Proof. We derive in L^μ.

$$\frac{\dfrac{\Delta\to\Pi}{\vdots}\ (\bullet\to)}{(\bullet\Delta)\to\Pi}\ (\bullet\to)$$

According to Lemma 6.1 $L\vdash\Gamma\to(\bullet\Delta)$ and $L\vdash(\bullet\Delta)\to(\bullet\Pi)$. By an application of cut we obtain $L\vdash\Gamma\to(\bullet\Pi)$, whence $L^\mu\vdash\Gamma\to\Pi$. ■

Lemma 6.3 *The rule $(\bullet\to)$ is reversible in L^μ.*

6.3 Definition of weights

For any sequence $\Gamma \in \mathrm{Tp}(m)$ we denote by $\lfloor\Gamma\rfloor$ the sequence of primitive types obtained from Γ by omitting parentheses and connectives. Thus $\lfloor\Gamma\rfloor$ is a word in the alphabet $\{p_1, p_2, p_3, \ldots\}$. Note that $\|\Gamma\| = |\lfloor\Gamma\rfloor|$.

Example 3

$$\lfloor p_1\ (p_1\backslash(p_2{\bullet}p_3))\rfloor = p_1p_1p_2p_3$$

We are going to associate with every L^μ-derivation $\Gamma \overset{\mathcal{D}}{\to} C_1 \ldots C_n$ a fragmentation of $\lfloor\Gamma\rfloor$ into n continuous subwords $\zeta_1, \ldots, \zeta_n$ such that $\lfloor\Gamma\rfloor = \zeta_1 \circ \ldots \circ \zeta_n$, where $\circ$ denotes concatenation of sequences of types. This will be done by induction on the length of the derivation $\mathcal{D}$. The *weight* of the type C_i (where $1 \le i \le n$) w.r.t. the derivation $\Gamma \overset{\mathcal{D}}{\to} C_1 \ldots C_n$ is the length of ζ_i and it is denoted by $\mathtt{w}^{\mathcal{D}}(C_i)$.

Obviously, the words $\zeta_1, \ldots, \zeta_n$ are uniquely defined if $\mathtt{w}^{\mathcal{D}}(C_1), \ldots, \mathtt{w}^{\mathcal{D}}(C_n)$ and Γ are given. The following definitions and lemmas are given in terms of $\mathtt{w}^{\mathcal{D}}(C_i)$, not ζ_i.

We shall write $\mathtt{w}^{\mathcal{D}}(C_iC_{i+1}\ldots C_j)$ for $\mathtt{w}^{\mathcal{D}}(C_i) + \mathtt{w}^{\mathcal{D}}(C_{i+1}) + \ldots + \mathtt{w}^{\mathcal{D}}(C_j)$.

Definition. The weights are defined by induction on the length of a derivation.

Case 1: Axiom $C_1 \ldots C_n \overset{\mathcal{D}}{\to} C_1 \ldots C_n$.

$\mathtt{w}^{\mathcal{D}}(C_i) \rightleftharpoons \|C_i\|$

Case 2: $(\to\backslash)$ $\quad \dfrac{A\Pi \to B}{\Pi \overset{\mathcal{D}}{\to} A\backslash B}\,(\to\backslash)$

$\mathtt{w}^{\mathcal{D}}(A\backslash B) \rightleftharpoons \|\Pi\|$

Case 3: $(\to/)$ $\quad \dfrac{\Pi A \to B}{\Pi \overset{\mathcal{D}}{\to} B/A}\,(\to/)$

$\mathtt{w}^{\mathcal{D}}(B/A) \rightleftharpoons \|\Pi\|$

Case 4: $(\backslash\to)$ $\quad \dfrac{\Phi \overset{\tilde{\mathcal{D}}}{\to} A \quad \Gamma B\Delta \overset{\hat{\mathcal{D}}}{\to} C_1 \ldots C_n}{\Gamma\Phi(A\backslash B)\Delta \overset{\mathcal{D}}{\to} C_1 \ldots C_n}\,(\backslash\to)$

$$\mathtt{w}^{\mathcal{D}}(C_i) \rightleftharpoons \begin{cases} \mathtt{w}^{\hat{\mathcal{D}}}(C_i) + \|\Phi\| + \|A\| & \text{if } \mathtt{w}^{\hat{\mathcal{D}}}(C_1 \ldots C_{i-1}) \le \|\Gamma\| \text{ and } \mathtt{w}^{\hat{\mathcal{D}}}(C_1 \ldots C_i) > \|\Gamma\| \\ \mathtt{w}^{\hat{\mathcal{D}}}(C_i) & \text{otherwise} \end{cases}$$

Case 5: $(/\to)$ $\quad \dfrac{\Phi \overset{\tilde{\mathcal{D}}}{\to} A \quad \Gamma B\Delta \overset{\hat{\mathcal{D}}}{\to} C_1 \ldots C_n}{\Gamma(B/A)\Phi\Delta \overset{\mathcal{D}}{\to} C_1 \ldots C_n}\,(/\to)$

$$\mathtt{w}^{\mathcal{D}}(C_i) \rightleftharpoons \begin{cases} \mathtt{w}^{\hat{\mathcal{D}}}(C_i) + \|\Phi\| + \|A\| & \text{if } \mathtt{w}^{\hat{\mathcal{D}}}(C_{i+1} \ldots C_n) \le \|\Delta\| \text{ and } \mathtt{w}^{\hat{\mathcal{D}}}(C_i \ldots C_n) > \|\Delta\| \\ \mathtt{w}^{\hat{\mathcal{D}}}(C_i) & \text{otherwise} \end{cases}$$

Case 6: $(\to\bullet)$ $$\frac{\Gamma \overset{\hat{\mathcal{D}}}{\to} \Theta A B \Xi}{\Gamma \overset{\mathcal{D}}{\to} \Theta(A\bullet B)\Xi}\ (\to\bullet)$$

$$\begin{aligned} \mathtt{w}^{\mathcal{D}}(A\bullet B) &\rightleftharpoons \mathtt{w}^{\hat{\mathcal{D}}}(A) + \mathtt{w}^{\hat{\mathcal{D}}}(B) \\ \mathtt{w}^{\mathcal{D}}(C) &\rightleftharpoons \mathtt{w}^{\hat{\mathcal{D}}}(C) \ \text{ for any placed type } C \text{ in } \Theta \text{ or } \Xi \end{aligned}$$

Case 7: $(\bullet\to)$ $$\frac{\Gamma A B \Delta \overset{\hat{\mathcal{D}}}{\to} \Theta}{\Gamma(A\bullet B)\Delta \overset{\mathcal{D}}{\to} \Theta}\ (\bullet\to)$$

$\mathtt{w}^{\mathcal{D}}(C) \rightleftharpoons \mathtt{w}^{\hat{\mathcal{D}}}(C)$ for any placed type C in Θ

Case 8: (CON) $$\frac{\Gamma \overset{\tilde{\mathcal{D}}}{\to} A_1 \dots A_n \quad \Delta \overset{\hat{\mathcal{D}}}{\to} B_1 \dots B_m}{\Gamma\Delta \overset{\mathcal{D}}{\to} A_1 \dots A_n B_1 \dots B_m}\ (CON)$$

$$\begin{aligned} \mathtt{w}^{\mathcal{D}}(A_i) &\rightleftharpoons \mathtt{w}^{\tilde{\mathcal{D}}}(A_i) \\ \mathtt{w}^{\mathcal{D}}(B_i) &\rightleftharpoons \mathtt{w}^{\hat{\mathcal{D}}}(B_i) \end{aligned}$$

Lemma 6.4 *If* $L^\mu \vdash \Gamma \overset{\mathcal{D}}{\to} C_1 \dots C_n$, *then* $\mathtt{w}^{\mathcal{D}}(C_i) > 0$ *for every* $i \leq n$.

Lemma 6.5 *If* $L^\mu \vdash \Gamma \overset{\mathcal{D}}{\to} C_1 \dots C_n$ *then* $\mathtt{w}^{\mathcal{D}}(C_1 \dots C_n) = \|\Gamma\|$.

Proof. Straightforward induction on the length of the proof $\mathcal{D}$. The only non-trivial rules are $(\backslash\to)$ and $(/\to)$. We consider the following case.

$$\frac{\Phi \overset{\tilde{\mathcal{D}}}{\to} A \quad \Gamma B \Delta \overset{\hat{\mathcal{D}}}{\to} C_1 \dots C_n}{\Gamma\Phi(A\backslash B)\Delta \overset{\mathcal{D}}{\to} C_1 \dots C_n}\ (\backslash\to)$$

By the induction hypothesis $\mathtt{w}^{\hat{\mathcal{D}}}(C_1 \dots C_n) = \|\Gamma B \Delta\| > \|\Gamma\|$. Consequently there exists a unique number i_0 such that $\mathtt{w}^{\hat{\mathcal{D}}}(C_1 \dots C_{i-1}) \leq \|\Gamma\|$ and $\mathtt{w}^{\hat{\mathcal{D}}}(C_1 \dots C_i) > \|\Gamma\|$. Evidently

$$\mathtt{w}^{\mathcal{D}}(C_1 \dots C_n) = \mathtt{w}^{\hat{\mathcal{D}}}(C_1 \dots C_n) + \|\Phi\| + \|A\| = \|\Gamma B \Delta\| + \|\Phi\| + \|A\|.$$

■

6.4 Properties of weights

The aim of this section is to prove two properties (Lemma 6.8 and Lemma 6.9 (i)), which will later be used in the proof of Lemma 3.1.

Lemma 6.6 *If* $\Pi\Pi' \overset{\mathcal{D}''}{\to} \Theta\Theta'$, $\Theta \neq \Lambda$, $\Theta' \neq \Lambda$, *and* $\mathtt{w}^{\mathcal{D}''}(\Theta) = \|\Pi\|$, *then*

(i) *there is a derivation* $\mathcal{D}$ *of the sequent* $\Pi \to \Theta$ *such that* $\mathtt{w}^{\mathcal{D}}(C) = \mathtt{w}^{\mathcal{D}''}(C)$ *for any placed type* C *in* Θ,

(ii) *there is a derivation* $\mathcal{D}'$ *of the sequent* $\Pi' \to \Theta'$ *such that* $\mathtt{w}^{\mathcal{D}'}(C) = \mathtt{w}^{\mathcal{D}''}(C)$ *for any placed type* C *in* Θ'.

Proof. Induction on the length of $\mathcal{D}''$.

Case 1: $$\frac{\Phi \overset{\hat{\mathcal{D}}}{\to} A \quad \Gamma B \Delta \overset{\tilde{\mathcal{D}}}{\to} C_1 \ldots C_n}{\Gamma\Phi(A\backslash B)\Delta \overset{\mathcal{D}''}{\to} C_1 \ldots C_n}\ (\backslash\to)$$

Note that, for all k, either $\mathtt{w}^{\mathcal{D}''}(C_1 \ldots C_k) \leq \|\Gamma\|$ or $\mathtt{w}^{\mathcal{D}''}(C_1 \ldots C_k) > \|\Gamma\| + \|\Phi\| + \|A\|$. Thus we have two subcases.

Case 1a: $$\frac{\Phi' \overset{\hat{\mathcal{D}}}{\to} A' \quad \Gamma\Gamma' B' \Delta' \overset{\tilde{\mathcal{D}}}{\to} \Theta\Theta'}{\Gamma\Gamma'\Phi'(A'\backslash B')\Delta' \overset{\mathcal{D}''}{\to} \Theta\Theta'}\ (\backslash\to) \quad \text{and } \mathtt{w}^{\tilde{\mathcal{D}}}(\Theta) = \|\Gamma\|$$

Case 1b: $$\frac{\Phi \overset{\hat{\mathcal{D}}}{\to} A \quad \Gamma B \Delta\Delta' \overset{\tilde{\mathcal{D}}}{\to} \Theta\Theta'}{\Gamma\Phi(A\backslash B)\Delta\Delta' \overset{\mathcal{D}''}{\to} \Theta\Theta'}\ (\backslash\to) \quad \text{and } \mathtt{w}^{\tilde{\mathcal{D}}}(\Theta) = \|\Gamma B \Delta\|$$

Both subcases are easily reduced to the induction hypothesis.

The rule $(/\to)$ is treated similarly, other cases are trivial. ■

Lemma 6.7 *If* $L^\mu \vdash \Pi_1 C \Delta_1 \overset{\mathcal{D}_1}{\to} \Theta_1\Xi_1$, $L^\mu \vdash \Pi_2 C \Delta_2 \overset{\mathcal{D}_2}{\to} \Theta_2\Xi_2$, *and*
$0 < \mathtt{w}^{\mathcal{D}_1}(\Theta_1) - \|\Pi_1\| = \mathtt{w}^{\mathcal{D}_2}(\Theta_2) - \|\Pi_2\| < \|C\|$,
then there is a derivation $\mathcal{D}$ *of the sequent* $\Pi_1 C \Delta_2 \to \Theta_1 \Xi_2$ *such that*

(i) $\mathtt{w}^{\mathcal{D}}(B) = \mathtt{w}^{\mathcal{D}_1}(B)$ *for any placed type* B *in* Θ_1;

(ii) $\mathtt{w}^{\mathcal{D}}(B) = \mathtt{w}^{\mathcal{D}_2}(B)$ *for any placed type* B *in* Ξ_2.

Proof. Induction on the total length of $\mathcal{D}_1$ and $\mathcal{D}_2$. Neither $\mathcal{D}_1$ nor $\mathcal{D}_2$ can be axiomatic. We distinguish three cases.
Case 1: C is the main type of the last rules of both $\mathcal{D}_1$ and $\mathcal{D}_2$.

Case 1a: $C = E\backslash F$
Given

$$\frac{\Phi_1 \overset{\hat{\mathcal{D}}_1}{\to} E \quad \Gamma_1 F \Delta_1 \overset{\tilde{\mathcal{D}}_1}{\to} \Theta_1\Xi_1}{\Gamma_1\Phi_1(E\backslash F)\Delta_1 \overset{\mathcal{D}_1}{\to} \Theta_1\Xi_1}\ (\backslash\to) \qquad \frac{\Phi_2 \overset{\hat{\mathcal{D}}_2}{\to} E \quad \Gamma_2 F \Delta_2 \overset{\tilde{\mathcal{D}}_2}{\to} \Theta_2\Xi_2}{\Gamma_2\Phi_2(E\backslash F)\Delta_2 \overset{\mathcal{D}_2}{\to} \Theta_2\Xi_2}\ (\backslash\to)$$

Here $\Pi_1 = \Gamma_1\Phi_1$ and $\Pi_2 = \Gamma_2\Phi_2$.

First we verify that $0 < \mathtt{w}^{\tilde{\mathcal{D}}_1}(\Theta_1) - \|\Gamma_1\| = \mathtt{w}^{\tilde{\mathcal{D}}_2}(\Theta_2) - \|\Gamma_2\| < \|F\|$ and next we apply the induction hypothesis to $\tilde{\mathcal{D}}_1$ and $\tilde{\mathcal{D}}_2$.

If $\mathtt{w}^{\tilde{\mathcal{D}}_1}(\Theta_1) \leq \|\Gamma_1\|$ then, by the definition of the weights, $\mathtt{w}^{\mathcal{D}_1}(\Theta_1) \leq \|\Gamma_1\| < \|\Pi_1\|$. This is in contradiction with $0 < \mathtt{w}^{\mathcal{D}_1}(\Theta_1) - \|\Pi_1\|$. Thus $\mathtt{w}^{\tilde{\mathcal{D}}_1}(\Theta_1) > \|\Gamma_1\|$.

Further, by the definition of weights, $\mathtt{w}^{\mathcal{D}_1}(\Theta_1) = \mathtt{w}^{\tilde{\mathcal{D}}_1}(\Theta_1) + \|\Phi_1\| + \|E\|$.
Therefore

$$\mathtt{w}^{\tilde{\mathcal{D}}_1}(\Theta_1) - \|\Gamma_1\| = (\mathtt{w}^{\mathcal{D}_1}(\Theta_1) - \|\Phi_1\| - \|E\|) - \|\Gamma_1\| = (\mathtt{w}^{\mathcal{D}_1}(\Theta_1) - \|\Gamma_1\Phi_1\|) - \|E\|$$

and simlarly

$$\mathtt{w}^{\tilde{\mathcal{D}}_2}(\Theta_2) - \|\Gamma_2\| = (\mathtt{w}^{\mathcal{D}_2}(\Theta_2) - \|\Gamma_2\Phi_2\|) - \|E\|.$$

By the assumption of the lemma, the right hand sides of these equalities are equal.
This proves $\mathtt{w}^{\tilde{\mathcal{D}}_1}(\Theta_1) - \|\Gamma_1\| = \mathtt{w}^{\tilde{\mathcal{D}}_2}(\Theta_2) - \|\Gamma_2\|$.

Now we see that

$$\mathtt{w}^{\tilde{\mathcal{D}}_1}(\Theta_1) - \|\Gamma_1\| = (\mathtt{w}^{\mathcal{D}_1}(\Theta_1) - \|\Gamma_1\Phi_1\|) - \|E\| < \|E\backslash F\| - \|E\| = \|F\|.$$

By the induction hypothesis we find a derivation $\tilde{\mathcal{D}}$ of the sequent $\Gamma_1 F\Delta_2 \to \Theta_1\Xi_2$.

$$\frac{\Phi_1 \overset{\hat{\mathcal{D}}_1}{\to} E \quad \Gamma_1 F \Delta_2 \overset{\tilde{\mathcal{D}}}{\to} \Theta_1\Xi_2}{\Gamma_1\Phi_1(E\backslash F)\Delta_2 \overset{\mathcal{D}}{\to} \Theta_1\Xi_2}\ (\backslash\to)$$

Case 1b: $C = E/F$
This case is treated in the same way as case 1a.
Case 1c: $C = E{\bullet}F$
Given

$$\frac{\Pi_1 EF\Delta_1 \overset{\hat{\mathcal{D}}_1}{\to} \Theta_1\Xi_1}{\Pi_1(E{\bullet}F)\Delta_1 \overset{\mathcal{D}_1}{\to} \Theta_1\Xi_1}\ ({\bullet}\to) \qquad \frac{\Pi_2 EF\Delta_2 \overset{\hat{\mathcal{D}}_2}{\to} \Theta_2\Xi_2}{\Pi_2(E{\bullet}F)\Delta_2 \overset{\mathcal{D}_2}{\to} \Theta_2\Xi_2}\ ({\bullet}\to)$$

If $\mathtt{w}^{\mathcal{D}_1}(\Theta_1) = \|\Pi_1\| + \|E\|$, then we find an appropriate derivation of $\Pi_1 EF\Delta_2 \to \Theta_1\Xi_2$ from Lemma 6.6 applying the rule (CON), otherwise from the induction hypothesis of this lemma. After that we derive

$$\frac{\Pi_1 EF\Delta_2 \to \Theta_1\Xi_2}{\Pi_1(E{\bullet}F)\Delta_2 \to \Theta_1\Xi_2}\ ({\bullet}\to)$$

CASE 2: C is not the main type of the last rule of $\mathcal{D}_1$.
We consider different subcases depending on the last rule of $\mathcal{D}_1$.
CASE 2a: $(\backslash\rightarrow)$

CASE 2a.i: Given

$$\dfrac{\Phi_1 \overset{\hat{\mathcal{D}}_1}{\rightarrow} E_1 \quad \Gamma_1 F_1 \Psi_1 C \Delta_1 \overset{\tilde{\mathcal{D}}_1}{\rightarrow} \Theta_1 \Xi_1}{\Gamma_1 \Phi_1 (E_1 \backslash F_1) \Psi_1 C \Delta_1 \overset{\mathcal{D}_1}{\rightarrow} \Theta_1 \Xi_1} (\backslash\rightarrow) \qquad \Pi_2 C \Delta_2 \overset{\mathcal{D}_2}{\rightarrow} \Theta_2 \Xi_2$$

It follows immediately from the definition of weights that
$\mathtt{w}^{\tilde{\mathcal{D}}_1}(\Theta_1) - \|\Gamma_1 F_1 \Psi_1\| = \mathtt{w}^{\mathcal{D}_1}(\Theta_1) - \|\Gamma_1 \Phi_1 (E_1 \backslash F_1) \Psi_1\|$. Thus we can apply the induction hypothesis to $\tilde{\mathcal{D}}_1$ and $\mathcal{D}_2$.

$$\dfrac{\Phi_1 \overset{\hat{\mathcal{D}}_1}{\rightarrow} E_1 \quad \Gamma_1 F_1 \Psi_1 C \Delta_2 \overset{\tilde{\mathcal{D}}}{\rightarrow} \Theta_1 \Xi_2}{\Gamma_1 \Phi_1 (E_1 \backslash F_1) \Psi_1 C \Delta_2 \overset{\mathcal{D}}{\rightarrow} \Theta_1 \Xi_2} (\backslash\rightarrow)$$

Other subcases are similar to case 2a.i.
CASE 3: C is not the main type of the last rule of $\mathcal{D}_2$.
Similar to case 2. ■

Lemma 6.8 *If $L^\mu \vdash \Gamma \overset{\mathcal{D}_1}{\rightarrow} \Theta_1 \Xi_1$, $L^\mu \vdash \Gamma \overset{\mathcal{D}_2}{\rightarrow} \Theta_2 \Xi_2$, and $\mathtt{w}^{\mathcal{D}_1}(\Theta_1) = \mathtt{w}^{\mathcal{D}_2}(\Theta_2)$ then there is a derivation $\mathcal{D}$ of the sequent $\Gamma \rightarrow \Theta_1 \Xi_2$ such that*

(i) $\mathtt{w}^{\mathcal{D}}(B) = \mathtt{w}^{\mathcal{D}_1}(B)$ *for any placed type B in Θ_1;*

(ii) $\mathtt{w}^{\mathcal{D}}(B) = \mathtt{w}^{\mathcal{D}_2}(B)$ *for any placed type B in Ξ_2.*

PROOF. Let $\Gamma = A_1 \ldots A_n$. Take $k \rightleftharpoons \min\{i \mid \|A_1 \ldots A_i\| \geq \mathtt{w}^{\mathcal{D}_1}(\Theta_1)\}$.
CASE 1: $\|A_1 \ldots A_k\| > \mathtt{w}^{\mathcal{D}_1}(\Theta_1)$
Evidently $\|A_1 \ldots A_{k-1}\| < \mathtt{w}^{\mathcal{D}_1}(\Theta_1) = \mathtt{w}^{\mathcal{D}_2}(\Theta_2) < \|A_1 \ldots A_{k-1}\| + \|A_k\|$.

We apply Lemma 6.7 with $C = A_k$, $\Pi_1 = \Pi_2 = A_1 \ldots A_{k-1}$, and $\Delta_1 = \Delta_2 = A_{k+1} \ldots A_n$.
CASE 2: $\|A_1 \ldots A_k\| = \mathtt{w}^{\mathcal{D}_1}(\Theta_1)$
If $k = 0$ then take $\mathcal{D} = \mathcal{D}_2$. If $k = n$ then take $\mathcal{D} = \mathcal{D}_1$.

If $0 < k < n$, then $L^\mu \vdash A_1 \ldots A_k \rightarrow \Theta_1$, $L^\mu \vdash A_{k+1} \ldots A_n \rightarrow \Xi_1$, $L^\mu \vdash A_1 \ldots A_k \rightarrow \Theta_2$, and $L^\mu \vdash A_{k+1} \ldots A_n \rightarrow \Xi_2$ according to Lemma 6.6. Applying the rule (CON) we obtain $L^\mu \vdash A_1 \ldots A_k A_{k+1} \ldots A_n \rightarrow \Theta_1 \Xi_2$. ■

There are several cut rules admissible in L^μ. We are interested in the following rule.

$$\dfrac{\Gamma \rightarrow \Theta \Delta \Xi \quad \Delta \rightarrow \Psi}{\Gamma \rightarrow \Theta \Psi \Xi}$$

Lemma 6.9 (i) *If* $L^\mu \vdash \Gamma \overset{\mathcal{D}_1}{\to} \Theta\Delta\Xi$ *and* $L^\mu \vdash \Delta \overset{\tilde{\mathcal{D}}}{\to} \Phi$, *then there is a derivation* $\mathcal{D}$ *of the sequent* $\Gamma\to\Theta\Phi\Xi$ *such that* $\mathtt{w}^{\mathcal{D}}(A) = \mathtt{w}^{\mathcal{D}_1}(A)$ *for any placed type* A *in* Θ *and* $\mathtt{w}^{\mathcal{D}}(B) = \mathtt{w}^{\mathcal{D}_1}(B)$ *for any placed type* B *in* Ξ.

(ii) *If* $L^\mu \vdash \Gamma_1 \overset{\mathcal{D}_1}{\to} \Theta\Delta_1$, $L^\mu \vdash \Gamma_3 \overset{\mathcal{D}_3}{\to} \Delta_3\Xi$, *and* $L^\mu \vdash \Delta_1\Delta_3 \overset{\tilde{\mathcal{D}}}{\to} \Phi$, *where* Δ_1 *and* Δ_3 *are non-empty, then there is a derivation* $\mathcal{D}$ *of the sequent* $\Gamma_1\Gamma_3\to\Theta\Phi\Xi$ *such that* $\mathtt{w}^{\mathcal{D}}(A) = \mathtt{w}^{\mathcal{D}_1}(A)$ *for any placed type* A *in* Θ *and* $\mathtt{w}^{\mathcal{D}}(B) = \mathtt{w}^{\mathcal{D}_3}(B)$ *for any placed type* B *in* Ξ.

(iii) *If* $L^\mu \vdash \Gamma_1 \overset{\mathcal{D}_1}{\to} \Theta\Delta_1$, $L^\mu \vdash \Gamma_2 \overset{\mathcal{D}_2}{\to} \Delta_2$, $L^\mu \vdash \Gamma_3 \overset{\mathcal{D}_3}{\to} \Delta_3\Xi$, *and* $L^\mu \vdash \Delta_1\Delta_2\Delta_3 \overset{\tilde{\mathcal{D}}}{\to} \Phi$, *where* Δ_1 *and* Δ_3 *are non-empty, then there is a derivation* $\mathcal{D}$ *of the sequent* $\Gamma_1\Gamma_2\Gamma_3\to\Theta\Phi\Xi$ *such that* $\mathtt{w}^{\mathcal{D}}(A) = \mathtt{w}^{\mathcal{D}_1}(A)$ *for any placed type* A *in* Θ *and* $\mathtt{w}^{\mathcal{D}}(B) = \mathtt{w}^{\mathcal{D}_3}(B)$ *for any placed type* B *in* Ξ.

Proof. First we prove (iii) by induction on the total length of $\mathcal{D}_1$ and $\mathcal{D}_3$. After this it is easy to prove (ii) and (i) in the similar way.

We consider a number of cases depending on the last rules of $\mathcal{D}_1$ and $\mathcal{D}_3$ in (iii).

Case 1: (CON) in $\mathcal{D}_1$

Case 1a:

$$\dfrac{\dfrac{\Gamma_1\to\Theta\Delta_1 \quad \Gamma_1'\to\Delta_1'}{\Gamma_1\Gamma_1'\to\Theta\Delta_1\Delta_1'}(CON) \quad \Gamma_2\to\Delta_2 \quad \Gamma_3\to\Delta_3\Xi \quad \Delta_1\Delta_2\Delta_3\to\Phi}{\Gamma_1\Gamma_1'\Gamma_2\Gamma_3\to\Theta\Phi\Xi}$$

$$\Downarrow$$

$$\dfrac{\Gamma_1\to\Theta\Delta_1 \quad \dfrac{\Gamma_1'\to\Delta_1' \quad \Gamma_2\to\Delta_2}{\Gamma_1'\Gamma_2\to\Delta_1'\Delta_2}(CON) \quad \Gamma_3\to\Delta_3\Xi \quad \Delta_1\Delta_2\Delta_3\to\Phi}{\Gamma_1\Gamma_1'\Gamma_2\Gamma_3\to\Theta\Phi\Xi}$$

Case 1b:

$$\dfrac{\dfrac{\Gamma_1'\to\Theta' \quad \Gamma_1\to\Theta\Delta_1}{\Gamma_1'\Gamma_1\to\Theta'\Theta\Delta_1}(CON) \quad \Gamma_2\to\Delta_2 \quad \Gamma_3\to\Delta_3\Xi \quad \Delta_1\Delta_2\Delta_3\to\Phi}{\Gamma_1'\Gamma_1\Gamma_2\Gamma_3\to\Theta'\Theta\Phi\Xi}$$

$$\Downarrow$$

$$\dfrac{\Gamma_1'\to\Theta' \quad \dfrac{\Gamma_1\to\Theta\Delta_1 \quad \Gamma_2\to\Delta_2 \quad \Gamma_3\to\Delta_3\Xi \quad \Delta_1\Delta_2\Delta_3\to\Phi}{\Gamma_1\Gamma_2\Gamma_3\to\Theta\Phi\Xi}}{\Gamma_1'\Gamma_1\Gamma_2\Gamma_3\to\Theta'\Theta\Phi\Xi}(CON)$$

Case 2: $(\backslash\to)$ in $\mathcal{D}_1$

$$\dfrac{\dfrac{\Psi\to A \quad \Gamma B\Pi\to\Theta\Delta_1}{\Gamma\Psi(A\backslash B)\Pi\to\Theta\Delta_1}(\backslash\to) \quad \Gamma_2\to\Delta_2 \quad \Gamma_3\to\Delta_3\Xi \quad \Delta_1\Delta_2\Delta_3\to\Phi}{\Gamma\Psi(A\backslash B)\Pi\Gamma_2\Gamma_3\to\Theta\Phi\Xi}$$

$$\Downarrow$$

$$\frac{\Psi\to A \qquad \dfrac{\Gamma B\Pi\to\Theta\Delta_1 \quad \Gamma_2\to\Delta_2 \quad \Gamma_3\to\Delta_3\Xi \quad \Delta_1\Delta_2\Delta_3\to\Phi}{\Gamma B\Pi\Gamma_2\Gamma_3\to\Theta\Phi\Xi}}{\Gamma\Psi(A\backslash B)\Pi\Gamma_2\Gamma_3\to\Theta\Phi\Xi}\,(\backslash\to)$$

Case 3: $(/\to)$ in $\mathcal{D}_1$

$$\frac{\dfrac{\Psi\to A \quad \Gamma B\Pi\to\Theta\Delta_1}{\Gamma(B/A)\Psi\Pi\to\Theta\Delta_1}\,(\backslash\to) \qquad \Gamma_2\to\Delta_2 \quad \Gamma_3\to\Delta_3\Xi \quad \Delta_1\Delta_2\Delta_3\to\Phi}{\Gamma(B/A)\Psi\Pi\Gamma_2\Gamma_3\to\Theta\Phi\Xi}$$

$$\Downarrow$$

$$\frac{\Psi\to A \qquad \dfrac{\Gamma B\Pi\to\Theta\Delta_1 \quad \Gamma_2\to\Delta_2 \quad \Gamma_3\to\Delta_3\Xi \quad \Delta_1\Delta_2\Delta_3\to\Phi}{\Gamma B\Pi\Gamma_2\Gamma_3\to\Theta\Phi\Xi}}{\Gamma(B/A)\Psi\Pi\Gamma_2\Gamma_3\to\Theta\Phi\Xi}\,(\backslash\to)$$

Case 4: $(\to\bullet)$ in $\mathcal{D}_1$

$$\frac{\dfrac{\Gamma_1\to\Theta\Psi AB\Pi}{\Gamma_1\to\Theta\Psi(A\bullet B)\Pi}\,(\to\bullet) \qquad \Gamma_2\to\Delta_2 \quad \Gamma_3\to\Delta_3\Xi \quad \Psi(A\bullet B)\Pi\Delta_2\Delta_3\to\Phi}{\Gamma_1\Gamma_2\Gamma_3\to\Theta\Phi\Xi}$$

$$\Downarrow$$

$$\frac{\Gamma_1\to\Theta\Psi AB\Pi \quad \Gamma_2\to\Delta_2 \quad \Gamma_3\to\Delta_3\Xi \quad \dfrac{\Psi AB\Pi\Delta_2\Delta_3\to\Phi}{\Psi(A\bullet B)\Pi\Delta_2\Delta_3\to\Phi}\,(\bullet\to)^{-1}}{\Gamma_1\Gamma_2\Gamma_3\to\Theta\Phi\Xi}$$

Case 5: $(\bullet\to)$ in $\mathcal{D}_1$
Similar to case 2.
Case 6: (CON), $(\backslash\to)$, $(/\to)$, $(\to\bullet)$, or $(\bullet\to)$ in $\mathcal{D}_3$
Similar to the corresponding case for $\mathcal{D}_1$.
Case 7: $(\to\backslash)$ or $(\to/)$ in $\mathcal{D}_1$ and $(\to\backslash)$ or $(\to/)$ in $\mathcal{D}_3$
Evidently, both Θ and Ξ are empty. According to Lemma 6.2, $L^\mu \vdash \Gamma_1\Gamma_2\Gamma_3\to\Phi$. ■

6.5 Construction of the R-models $\langle \mathbf{V}_\Gamma, \circ, v_\Gamma\rangle$

Proof of Lemma 3.1. We must construct a family of quasimodels $\langle \mathbf{V}_\Gamma \subset \mathbf{D}_\Gamma \times \mathbf{D}_\Gamma, \circ, v_\Gamma\rangle$ indexed by sequences of types $\Gamma \in \mathrm{Tp}^*$, such that $\langle \mathbf{V}_\Gamma, \circ\rangle \in \mathcal{S}_{\mathbf{Z}}$ for any Γ (cf. Example 1 (f)).

We have to point out designated elements $\psi \in \mathbf{D}_\Lambda$ and $\chi_\Gamma \in \mathbf{D}_\Gamma$ such that

(i) $(\forall\Gamma \in \mathrm{Tp}^*)$ $(\forall C \in \mathrm{Tp})$ $\langle\psi, \chi_\Gamma\rangle \in v_\Gamma(C) \Leftrightarrow L \vdash \Gamma\to C$

(ii) $(\forall\Gamma \in \mathrm{Tp}^*)$ $(\forall\Pi \in \mathrm{Tp}^*)$ $\mathbf{D}_\Gamma \subseteq \mathbf{D}_{\Gamma\Pi}$ and $\mathbf{V}_\Gamma \subseteq \mathbf{V}_{\Gamma\Pi}$

(iii) $(\forall\Gamma \in \mathrm{Tp}^*)$ $(\forall\Pi \in \mathrm{Tp}^*)$ $(\forall C \in \mathrm{Tp})$ $v_\Gamma(C) \subseteq v_{\Gamma\Pi}(C)$

(iv) $(\forall \Gamma \in \mathrm{Tp}^*)\ (\forall B \in \mathrm{Tp})\ \langle \chi_\Gamma, \chi_{\Gamma B}\rangle \in v_{\Gamma B}(B)$

This is done as follows.

$$\mathbf{D}_\Gamma \rightleftharpoons \{i \in \mathbf{N} \mid 0 \le i \le \|\Gamma\|\}$$

$$\mathbf{V}_\Gamma \rightleftharpoons \{\langle i,j\rangle \in \mathbf{N}\times\mathbf{N} \mid 0 \le i < j \le \|\Gamma\|\}$$

$$v_\Gamma(C) \rightleftharpoons$$

$$\{\langle i,j\rangle \in \mathbf{V}_\Gamma \mid (\exists \Theta \in \mathrm{Tp}^*)\ (\exists \Xi \in \mathrm{Tp}^*)\ (\exists \mathcal{D})\ L^\mu \vdash \Gamma \overset{\mathcal{D}}{\to} \Theta C \Xi,\ \mathtt{w}^{\mathcal{D}}(\Theta) = i,\ \mathtt{w}^{\mathcal{D}}(\Theta C) = j\}$$

$$\psi \rightleftharpoons 0 \qquad \chi_\Gamma \rightleftharpoons \|\Gamma\|$$

First, we verify that for any $\Gamma \in \mathrm{Tp}^*$, $\langle \mathbf{V}_\Gamma, \circ, v_\Gamma\rangle$ is a quasimodel.

(1) $v_\Gamma(A) \circ v_\Gamma(B) \subseteq v_\Gamma(A{\bullet}B)$

Let $\langle i,j\rangle \in v_\Gamma(A)$ and $\langle j,k\rangle \in v_\Gamma(B)$. This means that $L^\mu \vdash \Gamma \overset{\mathcal{D}_1}{\to} \Theta_1 A \Xi_1$, $L^\mu \vdash \Gamma \overset{\mathcal{D}_2}{\to} \Theta_2 B \Xi_2$, $\mathtt{w}^{\mathcal{D}_1}(\Theta_1) = i$, $\mathtt{w}^{\mathcal{D}_1}(\Theta_1 A) = j$, $\mathtt{w}^{\mathcal{D}_2}(\Theta_2) = j$, and $\mathtt{w}^{\mathcal{D}_2}(\Theta_2 B) = k$. According to Lemma 6.8, $L^\mu \vdash \Gamma \overset{\mathcal{D}}{\to} \Theta_1 A B \Xi_2$. Further, $L^\mu \vdash \Gamma \overset{\mathcal{D}'}{\to} \Theta_1 (A{\bullet}B) \Xi_2$. Note that $\mathtt{w}^{\mathcal{D}'}(\Theta_1) = i$ and $\mathtt{w}^{\mathcal{D}'}(\Theta_1(A{\bullet}B)) = k$. Thus $\langle i,k\rangle \in v_\Gamma(A{\bullet}B)$.

(1) $v_\Gamma(A{\bullet}B) \subseteq v_\Gamma(A) \circ v_\Gamma(B)$

Let $\langle i,k\rangle \in v_\Gamma(A{\bullet}B)$. This means that $L^\mu \vdash \Gamma \overset{\mathcal{D}_1}{\to} \Theta(A{\bullet}B)\Xi$, $\mathtt{w}^{\mathcal{D}_1}(\Theta) = i$, $\mathtt{w}^{\mathcal{D}_1}(\Theta(A{\bullet}B)) = k$. Note that $L^\mu \vdash (A{\bullet}B) \to AB$. According to Lemma 6.9 (i) there is a derivation $\mathcal{D}$ such that $L^\mu \vdash \Gamma \overset{\mathcal{D}}{\to} \Theta A B \Xi$, $\mathtt{w}^{\mathcal{D}}(\Theta) = i$, $\mathtt{w}^{\mathcal{D}}(\Theta AB) = k$. Let $j \rightleftharpoons \mathtt{w}^{\mathcal{D}}(\Theta A)$. Evidently $\langle i,j\rangle \in v_\Gamma(A)$ and $\langle j,k\rangle \in v_\Gamma(B)$.

(2)

Let $L \vdash A \to B$ and $\langle i,j\rangle \in v_\Gamma(A)$. This means that $L^\mu \vdash \Gamma \overset{\mathcal{D}_1}{\to} \Theta A \Xi$, $\mathtt{w}^{\mathcal{D}_1}(\Theta) = i$, and $\mathtt{w}^{\mathcal{D}_1}(\Theta A) = j$. In view of Lemma 6.1, $L^\mu \vdash A \to B$. According to Lemma 6.9 (i) there is a derivation $\mathcal{D}$ such that $L^\mu \vdash \Gamma \overset{\mathcal{D}}{\to} \Theta B \Xi$, $\mathtt{w}^{\mathcal{D}}(\Theta) = i$, $\mathtt{w}^{\mathcal{D}}(\Theta B) = j$. Thus $\langle i,j\rangle \in v_\Gamma(B)$.

Now we verify that $\langle \mathbf{V}_\Gamma, \circ, v_\Gamma\rangle$ satisfies (i)–(iv).

(i)

Let $L \vdash \Gamma \to C$. Then $L^\mu \vdash \Gamma \overset{\mathcal{D}}{\to} C$ and $\mathtt{w}^{\mathcal{D}}(C) = \|\Gamma\|$, whence $\langle 0, \|\Gamma\|\rangle \in v_\Gamma(C)$.

For the converse, let $\langle 0, \|\Gamma\|\rangle \in v_\Gamma(C)$. Then $L^\mu \vdash \Gamma \overset{\mathcal{D}}{\to} \Theta C \Xi$, $\mathtt{w}^{\mathcal{D}}(\Theta) = 0$, and $\mathtt{w}^{\mathcal{D}}(\Theta C) = \|\Gamma\| = \mathtt{w}^{\mathcal{D}}(\Theta C \Xi)$. Thus $\Theta = \Lambda$ and $\Xi = \Lambda$.

(ii)

Obvious.

(iii)

Let $\langle i,j\rangle \in v_\Gamma(C)$, i.e., $L^\mu \vdash \Gamma \overset{\mathcal{D}}{\to} \Theta C \Xi$, $\mathtt{w}^{\mathcal{D}}(\Theta) = i$, and $\mathtt{w}^{\mathcal{D}}(\Theta C) = j$. Applying the rule (CON) we obtain $L^\mu \vdash \Gamma\Pi \overset{\mathcal{D}'}{\to} \Theta C \Xi \Pi$, $\mathtt{w}^{\mathcal{D}'}(\Theta) = i$, and $\mathtt{w}^{\mathcal{D}'}(\Theta C) = j$.

(iv)

There is a derivation $\mathcal{D}$ such that $L^\mu \vdash \Gamma B \overset{\mathcal{D}}{\to} \Gamma B$, $\mathtt{w}^{\mathcal{D}}(\Gamma) = \|\Gamma\| = \chi_\Gamma$, and $\mathtt{w}^{\mathcal{D}}(\Gamma B) = \|\Gamma B\| = \chi_{\Gamma B}$. Thus $\langle \chi_\Gamma, \chi_{\Gamma B}\rangle \in v_{\Gamma B}(B)$. ■

Acknowledgements

I would like to thank S. Artemov, L. Beklemishev, M. Kanovich, and N. Pankratiev for constructive discussions of matters treated here. I am deeply grateful to J. van Benthem, K. Došen, G. Dzhaparidze, M. Kanazawa, Y. Venema, and S. Vujošević for making valuable comments and pointing out misprints and imprecisions in previous versions of this paper. I wish to express my gratitude to W. Buszkowski for providing information about the history of the Lambek calculus.

References

[1] J. van Benthem. *Language in Action: Categories, Lambdas and Dynamic Logic*. North-Holland, Amsterdam, (Studies in Logic 130), 1991.

[2] W. Buszkowski. Undecidability of some logical extensions of Ajdukiewicz-Lambek calculus. *Studia Logica*, 37:59–64, 1978.

[3] W. Buszkowski. Compatibility of categorial grammar with an associated category system. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 28:229–238, 1982.

[4] W. Buszkowski. Some decision problems in the theory of syntactic categories. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 28:539–548, 1982.

[5] W. Buszkowski. Completeness Results for Lambek Syntactic Calculus. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 32:13–28, 1986.

[6] K. Došen. A Completeness Theorem for the Lambek Calculus of Syntactic Categories. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 31:235–241, 1985.

[7] K. Došen. A brief survey of frames for the Lambek calculus. Bericht 5-90, Konstanzer Berichte zur Logik und Wissenschaftstheorie, Universität Konstanz, 1990. To appear in *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*.

[8] J. Lambek. The mathematics of sentence structure. *American Mathematical Monthly*, 65(3):154–170, 1958.

[9] Sz. Mikulás. The Completeness of the Lambek Calculus with respect to Relational Semantics. ILLC Prepublication Series LP–92–03, Institute for Logic, Language and Computation, University of Amsterdam, 1992.

[10] H. Andreká and Sz. Mikulás. Lambek calculus and its relational semantics: completeness and incompleteness. J. Logic Language Inform., 3:1–37, 1994.

[11] N. Pankratiev. On the completeness of the Lambek calculus with respect to relativized relational semantics. J. Logic Language Inform., 3:233–246, 1994.

[12] M. Pentus. Lambek calculus is L-complete. ILLC Prepublication Series LP–93–14, Institute for Logic, Language and Computation, University of Amsterdam, 1993.

[13] M. Pentus. Language completeness of the Lambek calculus. Proc. 9th Annual IEEE Symp. on Logic in Computer Science, Paris, France, 487–496, 1994.