The Conjoinability Relation in Lambek Calculus and Linear Logic

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Abstract

In 1958 J. Lambek introduced a calculus L of syntactic types and defined an equivalence relation on types: "$x \equiv y$ means that there exists a sequence $x = x_1, \dots, x_n = y$ $(n \geq 1)$, such that $x_i \to x_{i+1}$ or $x_{i+1} \to x_i$ $(1 \leq i < n)$". He pointed out that $x \equiv y$ if and only if there is join z such that $x \to z$ and $y \to z$.

This paper gives an effective characterization of this equivalence for the Lambek calculi L and LP, and for the multiplicative fragments of Girard's and Yetter's linear logics. Moreover, for the non-directed Lambek calculus LP and the multiplicative fragment of Girard's linear logic, we present linear time algorithms deciding whether two types are equal, and finding a join for them if they are.

Introduction

In [4] Joachim Lambek introduced a calculus L for deriving reduction laws of syntactic types, and studied an equivalence relation on types defined as follows:
$a \sim b$ iff there is a natural number n and there exist types $c_1, \dots, c_n$ such that $a = c_1$, $b = c_n$ and $\forall i < n$ $c_i \Rightarrow c_{i+1}$ or $c_{i+1} \Rightarrow c_i$. He also proved that this notion coincides with another one which was later called conjoinability. In [1] J. van Benthem pointed out that the question of decidability of this relation was still open in 1991.

In this paper we shall show that the equivalence of types a and b can be characterized in terms of some decidable invariants earlier introduced as necessary conditions for derivability of a sequent $a \Rightarrow b$. These invariants are primitive type counts $\#_p$ in

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the non-directed calculus LP and the "geometric invariant" or "balance" in the directed calculus L [1].

Since LP and L are fragments of the ordinary linear logic LL [2] and the cyclic linear logic CLL [6] respectively, we shall also study similar equivalence relation in both these logics. Let MLL and $MCLL$ denote the multiplicative fragments of LL and CLL. Characterization of the equivalence in these fragments involves a new invariant defined by

$$\natural p = 0, \quad \natural(a{\bullet}b) = \natural a + \natural b, \quad \natural(a^{\perp}) = 1 - \natural a,$$

where $\bullet$ and $(\ )^{\perp}$ denote the linear conjunction and negation.

Main results of this paper are the following:

- $a \sim b$ in L iff the sequent $a \Rightarrow b$ is balanced (Section 4.1).
- $a \sim b$ in LP iff $\#_p a = \#_p b$ for all literals p (Section 4.2).
- $a \sim b$ in MLL iff $\natural a = \natural b$ and $\#_p a = \#_p b$ for all literals p (Section 4.3).
- $a \sim b$ in $MCLL$ iff $\natural a = \natural b$ and the sequent $a \Rightarrow b$ is balanced (Section 4.3).
- if $b \sim c$ in LP, then there exists a type d such that $LP \vdash b \Rightarrow d$, $LP \vdash c \Rightarrow d$, and $\mid d \mid \leq 3(\mid b \mid + \mid c \mid) + 2$ (Section 5.1).
- if $b \sim c$ in MLL, then there exists a formula d such that $LP \vdash b \Rightarrow d$, $LP \vdash c \Rightarrow d$, and $\mid d \mid \leq 3(\mid b \mid + \mid c \mid)$ (Section 5.2).

Here $\mid d \mid$ denotes the total number of occurrences of literals in the formula d.

Earlier proofs of the first four of these results have been presented in [5] and [3]. In this paper we give new and shorter proofs.

1 Preliminaries

1.1 Lambek calculus

In [4, p. 165] J. Lambek introduced a formal system for deriving reduction laws of syntactic types. We shall consider this system (denoted here by L) and some variants of it.

The language of the Lambek calculus includes a non-empty denumerable set **Atom**of literals for primitive types, and three binary connectives $\bullet$, $\backslash$, $/$, called *product*, *left implication* and *right implication*. For the purpose of readability we shall omit parentheses whenever product occurs in the scope of a connective $\backslash$ or $/$. For example, $p{\bullet}s/q{\bullet}r$ means $(p{\bullet}s)/(q{\bullet}r)$.

Let $p, q, r, \ldots$ stand for elements of **Atom**. The letters from the beginning of the alphabet $a, b, c, \ldots$ denote types[1] built from literals with the help of product, left implication, and right implication. Let capitals $X, Y, Z, \ldots$ range over finite sequences of types,

[1]Throughout this paper we shall use the terms "type" and "formula" as synonyms.

possibly empty sequences. The concatenation of sequences X and Y will be denoted by X, Y.

First, we introduce the Gentzen style sequent calculus L^*. Sequents are of the form $X \Rightarrow b$ where b is a type and X is a sequence of types. The order of types in X is essential.

The only axiom scheme is $b \Rightarrow b$, where b is any type. Rules of inference are the following:

$$\frac{X \Rightarrow a \quad Y\ ,\ b\ ,\ Z \Rightarrow c}{Y\ ,\ X\ ,\ a\backslash b\ ,\ Z \Rightarrow c}\ (\ \backslash \Rightarrow\ ) \qquad \frac{a, X \Rightarrow b}{X \Rightarrow a\backslash b}\ (\ \Rightarrow \backslash)$$

$$\frac{X \Rightarrow a \quad Y\ ,\ b\ ,\ Z \Rightarrow c}{Y\ ,\ b/a\ ,\ X\ ,\ Z \Rightarrow c}\ (\ /\ \Rightarrow\ ) \qquad \frac{X, a \Rightarrow b}{X \Rightarrow b/a}\ (\ \Rightarrow /\ )$$

$$\frac{X\ ,\ a\ ,\ b\ ,\ Y \Rightarrow c}{X\ ,\ a{\bullet}b\ ,\ Y \Rightarrow c}\ (\ \bullet \Rightarrow\ ) \qquad \frac{X \Rightarrow a \quad Y \Rightarrow b}{X, Y \Rightarrow a{\bullet}b}\ (\ \Rightarrow \bullet)$$

$$\frac{X \Rightarrow a \quad Y, a, Z \Rightarrow b}{Y, X, Z \Rightarrow b}\ (\ \mathrm{CUT}\ )$$

We write $L^* \vdash X \Rightarrow b$ for "the sequent $X \Rightarrow b$ is derivable in L^* ".

Given a formal system T, we shall write $a \overset{T}{\cong} b$ iff $T \vdash a \Rightarrow b$ and $T \vdash b \Rightarrow a$. It appears that if $b \overset{T}{\cong} c$ then replacing an instance of type b in a sequent by type c does not have any effect on derivability in T. In particular $(a{\bullet}b){\bullet}c \overset{L^*}{\cong} a{\bullet}(b{\bullet}c)$ and $(a\backslash b)/c \overset{L^*}{\cong} a\backslash(b/c)$ for any types a, b and c. So, we can omit parentheses in types $(a{\bullet}b){\bullet}c$ and $(a\backslash b)/c$.

Adding to L^* the permutation rule

$$\frac{X, a, b, Y \Rightarrow c}{X, b, a, Y \Rightarrow c}\ (\ \mathrm{P}\ )$$

we obtain the non-directed Lambek calculus L^*P. In L^*P we have $b\backslash c \overset{L^*P}{\cong} c/b$ for any types b and c, whence left and right implications fall into one connective, which is often denoted by $b{\rightarrow}c$.

The original calculi L and LP are obtained from L^* and L^*P respectively, by adding the constraint that all sequents must have non-empty antecedents. In other words, in L and LP the rules ($\Rightarrow \backslash$) and ($\Rightarrow /$) may be applied only if the sequence X is not empty.

Evidently the following inclusions hold:

$$\begin{array}{ccc} L & \subset & LP \\ \cap & & \cap \\ L^* & \subset & L^*P \end{array}$$

Now we shall define a notion of duality, which often allows to cut by half proofs about derivability in Lambek calculi.

$$
\begin{array}{rcl}
\text{dual }(p) & \rightleftharpoons & p \\
\text{dual }(a{\bullet}b) & \rightleftharpoons & \text{dual }(b){\bullet}\text{ dual }(a) \\
\text{dual }(a\backslash b) & \rightleftharpoons & \text{dual }(b)/\text{ dual }(a) \\
\text{dual }(a/b) & \rightleftharpoons & \text{dual }(b)\backslash\text{ dual }(a) \\
\text{dual }(a_1,\ldots,a_n \Rightarrow b) & \rightleftharpoons & \text{dual }(a_n),\ldots,\text{ dual }(a_1) \Rightarrow \text{ dual }(b)
\end{array}
$$

In any of the calculi considered here, a sequent is derivable if and only if its dual is derivable.

1.2 Multiplicative fragments of ordinary and cyclic linear logics

First we introduce MLL — the multiplicative fragment of ordinary (commutative) linear logic. We shall denote linear negation by $(\ )^\perp$, linear implication by $\multimap$, linear conjunction and disjunction by $\bullet$ and $+$, and corresponding units by $\mathbf{1}$ and $\mathbf{0}$.

The formulas of linear logic are defined as follows:

- $\mathbf{0}$ and $\mathbf{1}$ are formulas,
- if $p \in \mathbf{Atom}$, then p and $p^\perp$ are formulas,
- if a and b are formulas, then $a{\bullet}b$ and $a{+}b$ are also formulas.

We introduce linear implication and negation as abbreviations defined as:

$$a \multimap b \ \rightleftharpoons\ (a)^\perp{+}b$$

$$
\begin{array}{rcl}
(\mathbf{1})^\perp & \rightleftharpoons & \mathbf{0} \\
(\mathbf{0})^\perp & \rightleftharpoons & \mathbf{1} \\
(p^\perp)^\perp & \rightleftharpoons & p \\
(a{\bullet}b)^\perp & \rightleftharpoons & (a)^\perp{+}(b)^\perp \\
(a{+}b)^\perp & \rightleftharpoons & (a)^\perp{\bullet}(b)^\perp \\
(a\multimap b)^\perp & \rightleftharpoons & a{\bullet}(b)^\perp
\end{array}
$$

Derivable objects of linear logic are sequents $\Rightarrow a_1;\ldots;a_n$, where a_1 , … , a_n are formulas. We shall interpret $\Rightarrow a_1;\ldots;a_n$ as $a_1{+}\cdots{+}a_n$.

The axiom scheme is $\Rightarrow a^\perp; a$. The inference rules of MLL are the following:

$$\frac{\Rightarrow X\ ;\ a\ ;\ b\ ;\ Y}{\Rightarrow X\ ;\ a{+}b\ ;\ Y}\,(+) \qquad\qquad \frac{\Rightarrow X; a \qquad \Rightarrow b; Y}{\Rightarrow X\ ;\ a{\bullet}b\ ;\ Y}\,(\bullet)$$

$$\frac{\Rightarrow X; a \qquad \Rightarrow a^{\perp}\ ;\ Y}{\Rightarrow X; Y}\ (\mathrm{CUT}) \qquad\qquad \frac{\Rightarrow X; a; b; Y}{\Rightarrow X; b; a; Y}\ (\mathrm{P})$$

Now we shall consider $MCLL$ — the multiplicative fragment of the cyclic (non-commutative) linear logic presented by Yetter in [6]. The connectives are the same as in MLL except that in MCLL there are two linear implications \ and / instead of $\multimap$.

The formulas are defined in the same way as in MLL, but the abbreviations are different.

$$\begin{array}{rcl} a\backslash b & \rightleftharpoons & (a)^{\perp}{+}b \\ b/a & \rightleftharpoons & b{+}(a)^{\perp} \end{array}$$

$$\begin{array}{rcl} (\mathbf{1})^{\perp} & \rightleftharpoons & \mathbf{0} \\ (\mathbf{0})^{\perp} & \rightleftharpoons & \mathbf{1} \\ (p^{\perp})^{\perp} & \rightleftharpoons & p \\ (a{\bullet}b)^{\perp} & \rightleftharpoons & b^{\perp}{+}a^{\perp} \\ (a{+}b)^{\perp} & \rightleftharpoons & b^{\perp}{\bullet}a^{\perp} \\ (a\backslash b)^{\perp} & \rightleftharpoons & b^{\perp}{\bullet}a \\ (b/a)^{\perp} & \rightleftharpoons & a{\bullet}b^{\perp} \end{array}$$

Again, the axiom scheme is $\Rightarrow a^{\perp}; a$. The inference rules of $MCLL$ are the following:

$$\frac{\Rightarrow X\ ;\ a\ ;\ b\ ;\ Y}{\Rightarrow X\ ;\ a{+}b\ ;\ Y}\ ({+}) \qquad\qquad \frac{\Rightarrow X; a \qquad \Rightarrow b; Y}{\Rightarrow X\ ;\ a{\bullet}b\ ;\ Y}\ ({\bullet})$$

$$\frac{\Rightarrow X; a \qquad \Rightarrow a^{\perp}\ ;\ Y}{\Rightarrow X; Y}\ (\mathrm{CUT}) \qquad\qquad \frac{\Rightarrow X; Y}{\Rightarrow Y; X}\ (\mathrm{ROT})$$

Obviously $MCLL \subset MLL$.

It is convenient to have in linear logics a notion of entailment similar to $a \Rightarrow b$ in Lambek calculi. We shall write $MLL \vdash a \Rightarrow b$ iff $MLL \vdash\ \Rightarrow a^{\perp}; b$, and $MCLL \vdash a \Rightarrow b$ iff $MCLL \vdash\ \Rightarrow a^{\perp}; b$.

Again, if $b \cong c$, then the formulas b and c may be replaced by each other. Immediate verification shows that $a{\bullet}(b{\bullet}c) \cong (a{\bullet}b){\bullet}c$ and $a{+}(b{+}c) \cong (a{+}b){+}c$. Hence we may omit parentheses in these expressions. We shall also omit parentheses in $a{+}(b{\bullet}c)$, but not in $(a{+}b){\bullet}c$.

The phenomenon of duality is present in linear logics as well.

$$\begin{array}{rcl} \mathrm{dual}\ (p) & \rightleftharpoons & p \\ \mathrm{dual}\ (p^{\perp}) & \rightleftharpoons & p^{\perp} \end{array}$$

$$\begin{aligned} \text{dual } (a{\bullet}b) &\rightleftharpoons \text{dual } (b){\bullet} \text{ dual } (a) \\ \text{dual } (a{+}b) &\rightleftharpoons \text{dual } (b){+} \text{ dual } (a) \\ \text{dual } (\Rightarrow a_1 \; ; \; \ldots \; ; \; a_n) &\rightleftharpoons \;\Rightarrow \text{ dual } (a_n) \; ; \; \ldots \; ; \; \text{dual } (a_1) \end{aligned}$$

Cut-elimination holds in all the calculi considered in this paper (cf. [4] [1] [6]).

Remark. We note that $MCLL$ is a conservative extension of L^* and MLL is a conservative extension of L^*P. This is easily proved with the help of cut-elimination and the invariant $\natural$, which will be introduced in Section 2.4.

2 Soundness results

In this section we extend the definitions of balance and primitive type counts from Lambek calculi (cf. [1]) to the case of linear logic. We present balance in group-theoretic terms.

A numerical model, called the *count of negations*, is introduced. Both ordinary and cyclic linear logics are sound with respect to this additional invariant.

All the lemmas in this section are proved by straightforward induction. We omit the proofs of most of them.

2.1 Balance in Lambek calculus

Let FG stand for the free group generated by literals from **Atom**. We shall denote the unit of FG by Λ. There is a mapping from the Lambek calculus formulas to FG, associating to each formula b its algebraic interpretation $[\![b]\!]$, defined in the natural way:

$$\begin{aligned} [\![p]\!] &\rightleftharpoons p \\ [\![a{\bullet}b]\!] &\rightleftharpoons [\![a]\!] \cdot [\![b]\!] \\ [\![a\backslash b]\!] &\rightleftharpoons [\![a]\!]^{-1} \cdot [\![b]\!] \\ [\![b/a]\!] &\rightleftharpoons [\![b]\!] \cdot [\![a]\!]^{-1} \end{aligned}$$

Definition. A Lambek calculus sequent $a_1, \ldots, a_n \Rightarrow b$ is *balanced* iff $[\![a_1]\!] \cdots [\![a_n]\!] \overset{FG}{=} [\![b]\!]$, where $\overset{FG}{=}$ denotes the equality in the free group FG.

It is easy to see that this definition of balance is equivalent to the one given in [1].

Example 1 The sequent $p/q, q \Rightarrow p$ is balanced, since $[\![(p/q){\bullet}q]\!] \equiv pq^{-1}q$, $[\![p]\!] \equiv p$, and $pq^{-1}q \overset{FG}{=} p$.

Lemma 1 (cf. [1]) *If $L^* \vdash X \Rightarrow b$ then $X \Rightarrow b$ is balanced.*

PROOF. Induction on derivations. We omit the trivial cases of axioms and the rule $(\bullet \Rightarrow\ )$.

CASE $(\Rightarrow \bullet)$: If $[\![X]\!] \stackrel{FG}{=} [\![a]\!]$ and $[\![Y]\!] \stackrel{FG}{=} [\![b]\!]$ then $[\![X]\!][\![Y]\!] \stackrel{FG}{=} [\![a]\!][\![b]\!]$.
CASE $(\Rightarrow \backslash)$: Multiplying the equality $[\![a]\!][\![X]\!] \stackrel{FG}{=} [\![b]\!]$ by $[\![a]\!]^{-1}$ on the left, one obtains $[\![X]\!] \stackrel{FG}{=} [\![a]\!]^{-1}[\![b]\!]$ as desired.
CASE $(\backslash \Rightarrow\ )$: If $[\![X]\!] \stackrel{FG}{=} [\![a]\!]$ then $[\![X]\!][\![a]\!]^{-1} \stackrel{FG}{=} \Lambda$.
In turn, $[\![Y]\!][\![b]\!][\![Z]\!] \stackrel{FG}{=} [\![c]\!]$ entails $[\![Y]\!][\![X]\!][\![a]\!]^{-1}[\![b]\!][\![Z]\!] \stackrel{FG}{=} [\![c]\!]$.
CASE $(\Rightarrow /)$ and $(/\ \Rightarrow\ )$: These rules are treated similarly to their duals $(\Rightarrow \backslash)$ and $(\backslash \Rightarrow\ )$. ■

2.2 Balance in linear logic

The formulas of linear logic are mapped into FG as follows:

$$\begin{array}{rcl}
[\![\mathbf{1}]\!] & \rightleftharpoons & \Lambda \\
[\![\mathbf{0}]\!] & \rightleftharpoons & \Lambda \\
[\![p]\!] & \rightleftharpoons & p \\
[\![a^{\perp}]\!] & \rightleftharpoons & [\![a]\!]^{-1} \\
[\![a{\bullet}b]\!] & \rightleftharpoons & [\![a]\!] \cdot [\![b]\!] \\
[\![a{+}b]\!] & \rightleftharpoons & [\![a]\!] \cdot [\![b]\!] \\
[\![a\backslash b]\!] & \rightleftharpoons & [\![a]\!]^{-1} \cdot [\![b]\!] \\
[\![b/a]\!] & \rightleftharpoons & [\![b]\!] \cdot [\![a]\!]^{-1}
\end{array}$$

Definition. A sequent of the cyclic linear logic $\Rightarrow a_1\ ;\ \dots\ ;\ a_n$ is *balanced* iff $[\![a_1]\!] \cdot \ldots \cdot [\![a_n]\!] \stackrel{FG}{=} \Lambda$.

Lemma 2 (i) *If $MCLL \vdash\ \Rightarrow X$ then $\Rightarrow X$ is balanced.*

(ii) *In particular, if $MCLL \vdash a \Rightarrow b$ then $[\![a]\!] \stackrel{FG}{=} [\![b]\!]$.*

2.3 Primitive type counts

Definition. For any literal $p \in \mathbf{Atom}$, the *p-count* $(\#_p)$ is the following mapping from types to integer numbers.

$$\begin{array}{rcl}
\#_p p & \rightleftharpoons & 1 \\
\#_p q & \rightleftharpoons & 0 \quad \text{, if } p \text{ and } q \text{ are distinct literals} \\
\#_p(a{\bullet}b) & \rightleftharpoons & \#_p a + \#_p b \\
\#_p(a\backslash b) & \rightleftharpoons & \#_p b - \#_p a \\
\#_p(b/a) & \rightleftharpoons & \#_p b - \#_p a
\end{array}$$

Lemma 3 (cf. [1]) *If* $L^*P \vdash a_1, \dots, a_n \Rightarrow b$
then $\#_p a_1 + \cdots + \#_p a_n = \#_p b$ *for any* $p \in$ **Atom** .

Definition. For linear logic we define the *p-count* as follows:

$$
\begin{aligned}
\#_p\mathbf{0} &\rightleftharpoons 0\\
\#_p\mathbf{1} &\rightleftharpoons 0\\
\#_p p &\rightleftharpoons 1\\
\#_p q &\rightleftharpoons 0 \text{ , if } p \text{ and } q \text{ are distinct literals}\\
\#_p(a^\perp) &\rightleftharpoons -\#_p a\\
\#_p(a \bullet b) &\rightleftharpoons \#_p a + \#_p b\\
\#_p(a + b) &\rightleftharpoons \#_p a + \#_p b\\
\#_p(a \multimap b) &\rightleftharpoons \#_p b - \#_p a
\end{aligned}
$$

Lemma 4 (i) *If* $MLL \vdash\ \Rightarrow a_1 \ ; \ \dots \ ; \ a_n$
then $\#_p a_1 + \cdots + \#_p a_n = 0$ *for any* $p \in$ **Atom** .

(ii) *In particular, if* $MLL \vdash a \Rightarrow b$ *then* $\#_p a = \#_p b$ *for any* $p \in$ **Atom** .

2.4 The count of negations

Definition. The *count of negations* is the following mapping $\natural$ from formulas of linear logic to integer numbers.

$$
\begin{aligned}
\natural p &\rightleftharpoons 0\\
\natural p^\perp &\rightleftharpoons 1\\
\natural\mathbf{1} &\rightleftharpoons 0\\
\natural\mathbf{0} &\rightleftharpoons 1\\
\natural(a \bullet b) &\rightleftharpoons \natural a + \natural b\\
\natural(a + b) &\rightleftharpoons \natural a + \natural b - 1
\end{aligned}
$$

The following equalities are easy consequences of the definitions of the linear negation and implications.

$$\natural(a^\perp) \rightleftharpoons 1 - \natural a$$

$$\begin{aligned} \natural(a\multimap b) &\rightleftharpoons \natural b - \natural a \\ \natural(a\backslash b) &\rightleftharpoons \natural b - \natural a \\ \natural(a/b) &\rightleftharpoons \natural a - \natural b \end{aligned}$$

Lemma 5 (i) *If* $MLL \vdash \ \Rightarrow a_1 \ ; \ \ldots \ ; \ a_n$ *then* $\natural(a_1 + \cdots + a_n) = 0$.

(ii) *In particular, if* $MLL \vdash a \Rightarrow b$ *then* $\natural a = \natural b$.

3 Equivalent formulas

Definition. Given a formal system T, we say that two formulas a and b are *equivalent* in theory T (we write $a \overset{T}{\sim} b$) iff there is a natural number n and there exist formulas $c_1, \ldots, c_n$ such that $a = c_1$, $b = c_n$, and

$$T \vdash c_i \Rightarrow c_{i+1} \ \text{ or } \ T \vdash c_{i+1} \Rightarrow c_i \ \text{ for any } \ i < n.$$

Remark. In other words, $\overset{T}{\sim}$ is the reflexive, symmetric, transitive closure of the relation $\overset{T}{\leq}$, where $a \overset{T}{\leq} b$ means that the sequent $a \Rightarrow b$ is derivable in the theory T.

Definition. We say that type c is a *join* for a set of formulas $\{a_1, \ldots, a_n\}$ in a theory T iff $T \vdash a_i \Rightarrow c$ for any $i \leq n$.

Definition. (cf. [1]) Two formulas a and b are *conjoinable* in a theory T iff there is a join for $\{a, b\}$ in T.

Example 2 Let c and d be any types of the Lambek calculus. Then $c\backslash c$ and d/d are conjoinable in the pure Lambek calculus L.

The following derivations show that $c\backslash c \bullet d/d$ is a consequent of both $c\backslash c$ and d/d in L.

$$\dfrac{\dfrac{c \Rightarrow c \qquad \dfrac{\dfrac{c \Rightarrow c \quad d \Rightarrow d}{c\ ,\ d \Rightarrow c\bullet d}(\Rightarrow \bullet)}{c \Rightarrow c\bullet d/d}(\Rightarrow /)}{c\ ,\ c\backslash c \Rightarrow c\bullet d/d}(\backslash \Rightarrow)}{c\backslash c \Rightarrow c\backslash c\bullet d/d}(\Rightarrow \backslash) \qquad\qquad \dfrac{\dfrac{\dfrac{d \Rightarrow d \qquad \dfrac{c \Rightarrow c \quad d \Rightarrow d}{c\ ,\ d \Rightarrow c\bullet d}(\Rightarrow \bullet)}{c\ ,\ d/d\ ,\ d \Rightarrow c\bullet d}(/ \Rightarrow)}{c\ ,\ d/d \Rightarrow c\bullet d/d}(\Rightarrow /)}{d/d \Rightarrow c\backslash c\bullet d/d}(\Rightarrow \backslash)$$

The next two lemmas, belonging to J. Lambek [4], show that conjoinability in L coincides with the equivalence relation $\overset{L}{\sim}$.

Lemma 6 (Diamond property) *Let a and b be any types of L. Then the following two assertions are equivalent.*

(i) *There exists a type c such that $L \vdash a \Rightarrow c$ and $L \vdash b \Rightarrow c$, i.e. a and b are conjoinable in L.*

(ii) *There exists a type d such that $L \vdash d \Rightarrow a$ and $L \vdash d \Rightarrow b$.*

$$\begin{array}{ccccc} & & a & & \\ & \nearrow & & \searrow & \\ d & & & & c \\ & \searrow & & \nearrow & \\ & & b & & \end{array}$$

In other words, we can find any of the types c or d, indicated on the figure, if the other three types are given.

Proof. We give a proof slightly shorter than in [4].

Case (i) $\rightarrow$ (ii) : We put $d = (a/c){\bullet}c{\bullet}(c\backslash b)$.

$$\cfrac{\cfrac{c \Rightarrow c \qquad \cfrac{b \Rightarrow c \qquad a \Rightarrow a}{a/c\ ,\ b \Rightarrow a}\,(/\ \Rightarrow)}{a/c\ ,\ c\ ,\ c\backslash b \Rightarrow a}\,(\backslash \Rightarrow)}{(a/c){\bullet}c{\bullet}(c\backslash b) \Rightarrow a}\,(\bullet \Rightarrow)$$

The sequent $(a/c){\bullet}c{\bullet}(c\backslash b) \Rightarrow b$ is derived dually.

Case (ii) $\rightarrow$ (i) : We put $c = (d/a)\backslash d/(b\backslash d)$.

$$\cfrac{\cfrac{\cfrac{a \Rightarrow a \qquad \cfrac{d \Rightarrow b \qquad d \Rightarrow d}{d\ ,\ b\backslash d \Rightarrow d}\,(\backslash \Rightarrow)}{d/a\ ,\ a\ ,\ b\backslash d \Rightarrow d}\,(/\ \Rightarrow)}{a\ ,\ b\backslash d \Rightarrow (d/a)\backslash d}\,(\Rightarrow \backslash)}{a \Rightarrow (d/a)\backslash d/(b\backslash d)}\,(\Rightarrow /)$$

The sequent $b \Rightarrow (d/a)\backslash d/(b\backslash d)$ is derived similarly. ■

Lemma 7 *Two types a and b are equivalent if and only if the assertion* (i) *(and consequently also* (ii)*) from the previous lemma holds.*

Proof. The 'if' part is obvious. To prove the 'only if' part, we assume that there are n types $e_1, \dots, e_n$ such that $e_i \Rightarrow e_{i+1}$ or $e_{i+1} \Rightarrow e_i$ for any $i < n$. Now the assertion (i) of Lemma 6 follows from (ii) $\to$ (i) and

$$\frac{e_i \Rightarrow e_{i+1} \qquad e_{i+1} \Rightarrow e_{i+2}}{e_i \Rightarrow e_{i+2}} \text{(CUT)}$$

by induction on n. ■

Corollary 8 *A finite set of types has a join in L if and only if the types are pairwise conjoinable in L.*

Lemma 9 *The relation $\overset{L}{\sim}$ is a congruence on types.*

Proof. Follows immediately from the admissibility of the rules

$$\frac{a \Rightarrow b}{a \bullet c \Rightarrow b \bullet c} \qquad \frac{a \Rightarrow b}{c \backslash a \Rightarrow c \backslash b} \qquad \frac{a \Rightarrow b}{b \backslash c \Rightarrow a \backslash c}$$

■

Corollary 10 (Replacement property of the equivalence) *Let c_d be a type, containing a subtype d, and let c_b come from c_d by replacing one occurrence of d by a type b. If $d \sim b$, then $c_d \sim c_b$.*

4 Completeness of the equivalence with respect to group interpretations

In this section we prove that the notion of equivalence of two types in the directed Lambek calculus coincides with the balance, and in the undirected Lambek calculus it coincides with the equality of primitive type counts. Similar characterization of equivalence in linear logics involves in addition the count of negations.

4.1 Pure Lambek calculus

Theorem 1 *For any types a and b, the following three clauses are equivalent.*

(i) $a \overset{L}{\sim} b$

(ii) $a \overset{L^*}{\sim} b$

(iii) $[\![a]\!] \overset{FG}{=} [\![b]\!]$

Proof. (i) $\to$ (ii)
Follows from $L \subset L^*$.

(ii) $\to$ (iii)
Let $L^* \vdash a \Rightarrow c$ and $L^* \vdash b \Rightarrow c$. By Lemma 1 this entails $[\![a]\!] \overset{FG}{=} [\![c]\!]$ and $[\![b]\!] \overset{FG}{=} [\![c]\!]$. Hence $[\![a]\!] \overset{FG}{=} [\![b]\!]$.

(iii) $\to$ (i)
We prove that if $[\![a]\!] \overset{FG}{=} [\![b]\!]$ then $[a]_\sim = [b]_\sim$, where $[\ ]_\sim$ denotes the equivalence class of a type with respect to the relation $\overset{L}{\sim}$.

First, we note that all types of the form $c\backslash c$ and d/d belong to one equivalence class, which we shall denote by $\mathbf{1}_\sim$ (cf. Example 2).

Now we verify that the equivalence classes form a group with unit $\mathbf{1}_\sim$ and operators

$$[c]_\sim \cdot [d]_\sim \rightleftharpoons [c{\bullet}d]_\sim \ \ , \ \ [c]_\sim^{-1} \rightleftharpoons [c\backslash c/c]_\sim .$$

The associativity law is obvious. The property $[c]_\sim \cdot \mathbf{1}_\sim = [c]_\sim$ is evident from $L \vdash c{\bullet}(c\backslash c) \Rightarrow c$. Similarly, $[c]_\sim^{-1} \cdot [c]_\sim = \mathbf{1}_\sim$ follows from $L \vdash (c\backslash c/c){\bullet}c \Rightarrow c\backslash c$. The dual laws hold also. Hence the classes $[c]_\sim$ form a group.

Let us consider the mapping $[\![p]\!] \mapsto [p]_\sim$ from the generators of the free group FG into the group of the congruence classes of types. We extend this mapping to a homomorphism h. All that we need is to prove that, for any type c, h maps $[\![c]\!]$ to $[c]_\sim$. This is done by induction on the construction of the type c.

Case 1: $h([\![c{\bullet}d]\!]) = h([\![c]\!]) \cdot h([\![d]\!]) = [c]_\sim \cdot [d]_\sim = [c{\bullet}d]_\sim$

Case 2: $h([\![c\backslash d]\!]) = h([\![c]\!]^{-1} \cdot [\![d]\!]) = [c]_\sim^{-1} \cdot [d]_\sim$

It is sufficient to prove that $[c]_\sim^{-1} \cdot [d]_\sim = [c\backslash d]_\sim$. We have

$$(c\backslash(c/c)){\bullet}d \overset{L}{\sim} (c\backslash(d/d)){\bullet}d \overset{L}{\cong} ((c\backslash d)/d){\bullet}d.$$

Evidently $L \vdash ((c\backslash d)/d){\bullet}d \Rightarrow c\backslash d$. Hence $(c\backslash c/c){\bullet}d \overset{L}{\sim} c\backslash d$.

Case 3: Similarly to the previous case, we verify that $h([\![c/d]\!]) = [c/d]_\sim$. ■

4.2 Lambek calculus with permutation

Theorem 2 *For any types a and b, the following three clauses are equivalent.*

(i) $a \overset{LP}{\sim} b$

(ii) $a \overset{L^*P}{\sim} b$

(iii) $\#_p a = \#_p b$ *for any* $p \in \mathbf{Atom}$.

Proof. We note that "$\#_p a = \#_p b$ for any $p \in \mathbf{Atom}$" means that $[\![a]\!]$ and $[\![b]\!]$ are equal in the free Abelian group generated by **Atom**. Now the theorem is proved by a trivial modification of the proof of Theorem 1. ■

Remark. If we extend the language of the calculus L^* with the constant $\mathbf{1}$, add the axiom $\Rightarrow \mathbf{1}$ and the rule

$$\frac{X\ ,\ Y \Rightarrow a}{X\ ,\ \mathbf{1}\ ,\ Y \Rightarrow a}\ (\mathbf{1} \Rightarrow\ )$$

we obtain the calculus $L_{\mathbf{1}}$. In the similar way, $LP_{\mathbf{1}}$ is obtained from L^*P. Theorem 1 and Theorem 2 hold also in calculi $L_{\mathbf{1}}$ and $LP_{\mathbf{1}}$ respectively, if we extend the definitions of $[\![\ ]\!]$ and $\#_p$ by

$$[\![\mathbf{1}]\!] \rightleftharpoons \Lambda \text{ and } \#_p\mathbf{1} \rightleftharpoons 0.$$

4.3 Linear logics

In the full linear logic any two formulas a and b are trivially equivalent, because

$$a \Rightarrow a \oplus b \text{ and } b \Rightarrow a \oplus b\ \ ,$$

where $a \oplus b$ stands for the non-linear (additive) disjunction, sometimes also denoted by $a \sqcup b$. The situation is different in the multiplicative fragment.

Theorem 3 *Let a and b be any two formulas of $MCLL$. Then $a \overset{MCLL}{\sim} b$ if and only if $\natural a = \natural b$ and $[\![a]\!] \overset{FG}{=} [\![b]\!]$.*

PROOF. The 'if' part is an easy consequence of soundness with respect to $\natural$ and balance (cf. Lemma 5 (ii) and Lemma 2 (ii)).

To prove the 'only if' part we introduce a new group G, generated by elements of **Atom**together with $\mathbf{0}$, and satisfying the identity $\mathbf{0} \cdot c = c \cdot \mathbf{0}$ for any $c \in G$. Let Λ stand for the group unit of G. By $[\![\ ]\!]_G$ we denote the following mapping from the set of formulas of linear logic to the group G.

$$\begin{array}{rcl}
[\![\mathbf{1}]\!]_G & \rightleftharpoons & \Lambda \\
[\![\mathbf{0}]\!]_G & \rightleftharpoons & \mathbf{0} \\
[\![p]\!]_G & \rightleftharpoons & p \\
[\![c^\perp]\!]_G & \rightleftharpoons & [\![c]\!]_G^{-1} \cdot \mathbf{0} \\
[\![c \bullet d]\!]_G & \rightleftharpoons & [\![c]\!]_G \cdot [\![d]\!]_G \\
[\![c + d]\!]_G & \rightleftharpoons & [\![c]\!]_G \cdot [\![d]\!]_G \cdot \mathbf{0}^{-1} \\
[\![c \backslash d]\!]_G & \rightleftharpoons & [\![c]\!]_G^{-1} \cdot [\![d]\!]_G \\
[\![c / d]\!]_G & \rightleftharpoons & [\![c]\!]_G \cdot [\![d]\!]_G^{-1}
\end{array}$$

Obviously, $[\![a]\!]_G \overset{G}{=} [\![b]\!]_G$ if and only if $\natural a = \natural b$ and $[\![a]\!] \overset{FG}{=} [\![b]\!]$.

Similarly to Theorem 1, we prove that if $[\![a]\!]_G \overset{G}{=} [\![b]\!]_G$ then $[a]_\sim = [b]_\sim$, where $[\ ]_\sim$ denotes now the equivalence class with respect to the relation $\overset{MCLL}{\sim}$.

To extend the mapping $[\![p]\!]_G \mapsto [p]_\sim$ to a homomorphism we need two facts in addition to what has already been proved in Theorem 1.

(1) $[\mathbf{0}]_\sim \cdot [c]_\sim = [c]_\sim \cdot [\mathbf{0}]_\sim$

Evidently $MCLL \vdash c{\bullet}c^\perp \Rightarrow \mathbf{0}$ and $MCLL \vdash c^\perp{\bullet}c \Rightarrow \mathbf{0}$.

Hence $\mathbf{0}{\bullet}c \overset{MCLL}{\sim} (c{\bullet}c^\perp){\bullet}c \overset{MCLL}{\sim} c{\bullet}(c^\perp{\bullet}c) \overset{MCLL}{\sim} c{\bullet}\mathbf{0}$.

(2) $[c\backslash c/c]_\sim \cdot [\mathbf{0}]_\sim = [c^\perp]_\sim$

This follows from

$$(c\backslash c/c){\bullet}(c{\bullet}c^\perp) \overset{MCLL}{\sim} (((c\backslash c)/c){\bullet}c){\bullet}c^\perp \overset{MCLL}{\sim} (c\backslash c){\bullet}c^\perp \overset{MCLL}{\sim} c^\perp$$

■

Theorem 4 *Let a and b be any two formulas of MLL. Then $a \overset{MLL}{\sim} b$ if and only if $\natural a = \natural b$ and $\#_p a = \#_p b$ for any $p \in$* **Atom**.

PROOF. We note that $\natural a = \natural b$ and $\#_p a = \#_p b$ for any $p \in$ **Atom** if and only if $[\![a]\!]_G$ and $[\![b]\!]_G$ are equal in the free Abelian group. Hence the theorem is proved by a trivial modification of the proof of Theorem 3. ■

Remark. If we leave out the units **1** and **0** we obtain the constant-free fragments of $MCLL$ and MLL. Theorems 3 and 4 hold also for these fragments.

4.4 Complete description of derivability invariants

Theorem 5 *Let Δ be an arbitrary set. Suppose that a mapping ϕ from formulas to Δ is a MLL-derivability invariant, i.e., $\phi(a) = \phi(b)$ whenever $MLL \vdash a \Rightarrow b$. Then $\phi(b)$ is actually a function of $\#_p b$ and $\natural b$.*

PROOF. We have to show that, if $\natural a = \natural b$ and $\#_p a = \#_p b$ for any $p \in$ **Atom**, then $\phi(a) = \phi(b)$. This is an immediate consequence of the 'if' part of Theorem 4 and the definition of equivalent formulas. ■

Example 3 Let $\#_\bullet(b)$ denote the number of occurrences of the connective $\bullet$ in a formula b. Similarly, let $\#_+(b)$ count the occurrences of the connective $+$. It is easy to verify that $\phi(b) \rightleftharpoons \#_\bullet(b) - \#_+(b)$ is a derivability invariant for the calculus MLL.

According to Theorem 5, the function ϕ must be definable with the help of $\#_p$ and $\natural$. Really, straightforward induction shows that

$$\phi(b) = \sum_{p \in \mathbf{Atom}} \#_p(b) + 2 \cdot \natural(b) - 1.$$

Remark. Similar complete sets of derivability invariants in other calculi are

1. $[\![\ ]\!]$ for L and L^*,
2. $\#_p$ for LP and L^*P,
3. $[\![\ ]\!]$ and $\natural$ for $MCLL$.

5 Linear length joins

In this section we prove that in the Lambek calculus with permutation and in the ordinary linear logic any pair of equivalent formulas $a \sim b$ has a linear length join c such that $a \Rightarrow c$ and $b \Rightarrow c$.

We define the length of a formula b (denoted by $\mid b \mid$) as the total number of occurrences of literals and constants $\mathbf{0}$ and $\mathbf{1}$.

5.1 Lambek calculus with permutation LP

First we introduce special counts for positive and negative occurrences of literals.

$$
\begin{array}{rcl}
\#_p^+ p & \rightleftharpoons & 1 \\
\#_p^+ q & \rightleftharpoons & 0, \quad \text{where } p \text{ and } q \text{ are distinct literals} \\
\#_p^- q & \rightleftharpoons & 0, \quad \text{where } p \text{ and } q \text{ are any literals (possibly coinciding)} \\
\#_p^+ (a \bullet b) & \rightleftharpoons & \#_p^+ a + \#_p^+ b \\
\#_p^- (a \bullet b) & \rightleftharpoons & \#_p^- a + \#_p^- b \\
\#_p^+ (a \to b) & \rightleftharpoons & \#_p^- a + \#_p^+ b \\
\#_p^- (a \to b) & \rightleftharpoons & \#_p^+ a + \#_p^- b
\end{array}
$$

Evidently $\#_p a = \#_p^+ a - \#_p^- a$ for any type a and any literal p.

Let q be a fixed literal. Our aim is to show that any pair of equivalent types has a join of the form $(c_1 \bullet \ldots \bullet c_l \bullet q) \to q$, where, for any $i \leq l$, c_i is either $q \bullet (q \to p)$ or $p \bullet q \to q$ for some literal p.

Lemma 11 *Let a be any formula of LP containing only literals $p_1, \ldots, p_n$. Let $\#_{p_i}^+ a = k_i$ and $\#_{p_i}^- a = m_i$. Then*

(i) $LP \vdash (q \bullet (q \to p_1))^{k_1}, \ldots, (q \bullet (q \to p_n))^{k_n}, (p_1 \bullet q \to q)^{m_1}, \ldots, (p_n \bullet q \to q)^{m_n} \Rightarrow a,$

(ii) $LP \vdash a, (q \bullet (q \to p_1))^{m_1}, \ldots, (q \bullet (q \to p_n))^{m_n}, (p_1 \bullet q \to q)^{k_1}, \ldots, (p_n \bullet q \to q)^{k_n}, q \Rightarrow q.$

Here b^k denotes $\underbrace{b, \ldots, b}_{k \text{ times}}$.

Proof. Induction on the construction of the formula a.

Case 1: $a = p_i$

(i)

$$
\dfrac{\dfrac{q \Rightarrow q \qquad p_i \Rightarrow p_i}{q\ ,\ q \to p_i \Rightarrow p_i}\,(\to\ \Rightarrow)}{q \bullet (q \to p_i) \Rightarrow p_i}\,(\bullet\ \Rightarrow)
$$

(ii)

$$\cfrac{\cfrac{\cfrac{p_i \Rightarrow p_i \qquad q \Rightarrow q}{p_i\ ,\ q \Rightarrow p_i \bullet q}\ (\Rightarrow \bullet) \qquad q \Rightarrow q}{p_i\ ,\ q\ ,\ p_i \bullet q \to q \Rightarrow q}\ (\to \Rightarrow)}{p_i\ ,\ p_i \bullet q \to q\ ,\ q \Rightarrow q}\ (\mathrm{P})$$

CASE 2: $a = b \bullet c$

By induction hypothesis there are LP-derivable sequents

$$X \Rightarrow b \quad , \quad b, Y, q \Rightarrow q \quad , \quad Z \Rightarrow c \quad , \quad c, W, q \Rightarrow q$$

of the special form given in the formulation of the lemma.

(i)

$$\cfrac{X \Rightarrow b \qquad Z \Rightarrow c}{X\ ,\ Z \Rightarrow b \bullet c}\ (\Rightarrow \bullet)$$

(ii)

$$\cfrac{\cfrac{b\ ,\ Y\ ,\ q \Rightarrow q \qquad c\ ,\ W\ ,\ q \Rightarrow q}{b\ ,\ c\ ,\ Y\ ,\ W\ ,\ q \Rightarrow q}\ (\mathrm{CUT})}{b \bullet c\ ,\ Y\ ,\ W\ ,\ q \Rightarrow q}\ (\bullet \Rightarrow)$$

CASE 3: $a = b \to c$

Obviously there exists $i \leq n$ such that $\#^{+}_{p_i} c > 0$. Therefore we may write the induction hypothesis as

$$X \Rightarrow b \quad , \quad b, Y, q \Rightarrow q \quad , \quad q \bullet (q \to p_i), Z \Rightarrow c \quad , \quad c, W, q \Rightarrow q.$$

(i)

$$\cfrac{\cfrac{\cfrac{\cfrac{b\ ,\ Y\ ,\ q \Rightarrow q \qquad q \to p_i \Rightarrow q \to p_i}{b\ ,\ Y\ ,\ q\ ,\ q \to p_i \Rightarrow q \bullet (q \to p_i)}\ (\Rightarrow \bullet) \qquad q \bullet (q \to p_i)\ ,\ Z \Rightarrow c}{b\ ,\ Y\ ,\ q\ ,\ q \to p_i\ ,\ Z \Rightarrow c}\ (\mathrm{CUT})}{b\ ,\ Y\ ,\ q \bullet (q \to p_i)\ ,\ Z \Rightarrow c}\ (\bullet \Rightarrow)}{Y\ ,\ q \bullet (q \to p_i)\ ,\ Z \Rightarrow b \to c}\ (\Rightarrow \to)$$

(ii)

$$\cfrac{\cfrac{X \Rightarrow b \qquad c\ ,\ W\ ,\ q \Rightarrow q}{X\ ,\ b \to c\ ,\ W\ ,\ q \Rightarrow q}\ (\to \Rightarrow)}{b \to c\ ,\ X\ ,\ W\ ,\ q \Rightarrow q}\ (\mathrm{P})$$

■

Lemma 12 *If* $k' - k = m' - m \geq 0$ *and*

$$LP \vdash X \ , \ (q\bullet(q\to p))^k \ , \ (p\bullet q\to q)^m \Rightarrow q$$

then

$$LP \vdash X \ , \ (q\bullet(q\to p))^{k'} \ , \ (p\bullet q\to q)^{m'} \Rightarrow q.$$

Proof. Induction on $k' - k$.
For the induction step we verify that $LP \vdash q\bullet(q\to p) \ , \ p\bullet q\to q \ , \ X \Rightarrow q$ whenever $LP \vdash X \Rightarrow q$.

$$\dfrac{\dfrac{\dfrac{X \Rightarrow q \qquad \dfrac{\dfrac{p \Rightarrow p \qquad q \Rightarrow q}{p \ , \ q \Rightarrow p\bullet q}\,(\Rightarrow \bullet) \qquad q \Rightarrow q}{p \ , \ q \ , \ p\bullet q\to q \Rightarrow q}\,(\to \Rightarrow)}{X \ , \ q\to p \ , \ q \ , \ p\bullet q\to q \Rightarrow q}\,(\to \Rightarrow)}{X \ , \ q \ , \ q\to p \ , \ p\bullet q\to q \Rightarrow q}\,(\mathrm{P})}{X \ , \ q\bullet(q\to p) \ , \ p\bullet q\to q \Rightarrow q}\,(\bullet \Rightarrow)$$

■

Theorem 6 *If two types* a *and* b *are equivalent in* LP *then there is a type* c *such that*

$$LP \vdash a \Rightarrow c, \quad LP \vdash b \Rightarrow c \quad \textit{and} \quad |\, c \,| \leq 3 \,|\, a \,| + 3 \,|\, b \,| + 2.$$

Proof. We put

$$c = (q\bullet(q\to p_1))^{m'_1}\bullet \ldots \bullet(q\bullet(q\to p_n))^{m'_n}\bullet(p_1\bullet q\to q)^{k'_1}\bullet \ldots \bullet(p_n\bullet q\to q)^{k'_n}\bullet q\to q,$$

where $k'_i = \max(\#^+_{p_i}a, \#^+_{p_i}b)$ and $m'_i = \max(\#^-_{p_i}a, \#^-_{p_i}b)$.
Evidently $\#_{p_i}a = \#_{p_i}b$ implies $\#^+_{p_i}b - \#^+_{p_i}a = \#^-_{p_i}b - \#^-_{p_i}a$. Hence

$$k'_i - \#^+_{p_i}a = m'_i - \#^-_{p_i}a \geq 0.$$

Now $LP \vdash a \Rightarrow c$ follows from Lemma 11 (ii) by applying Lemma 12 n times. Similarly $LP \vdash b \Rightarrow c$.

Since

$$\sum_{i=1}^{n}(\#^+_{p_i}a + \#^-_{p_i}a) = |\, a \,|,$$

we have

$$\sum_{i=1}^{n}(k'_i + m'_i) \leq |\, a \,| + |\, b \,| \quad \text{and thus} \quad |\, c \,| \leq 3 \,|\, a \,| + 3 \,|\, b \,| + 2.$$

■

5.2 Linear logic MLL

Theorem 7 *For any pair of equivalent MLL-formulas b and c, there is a formula d such that*

$$MLL \vdash b \Rightarrow d, \quad MLL \vdash c \Rightarrow d, \quad \text{and} \quad |\, d \,| \leq 3(|\, b \,| + |\, c \,|).$$

SKETCH OF THE PROOF.
Let k_i denote the number of positive occurrences of p_i in the formula b, and k_i' stand for the number of negative occurrences. By l and l' we denote the numbers of occurrences of $\mathbf{0}$ in b and c respectively (we assume that $\mathbf{0}$ and $\mathbf{1}$ have no negative occurrences). We put

$$d = \sum_i (\underbrace{p_i \bullet \mathbf{0} + \ldots + p_i \bullet \mathbf{0}}_{k_i + k_i' \text{ times}} + \underbrace{p_i{}^{\perp} \bullet \mathbf{0} + \ldots + p_i{}^{\perp} \bullet \mathbf{0}}_{m_i + m_i' \text{ times}}) + \underbrace{\mathbf{0} \bullet \mathbf{0} + \ldots + \mathbf{0} \bullet \mathbf{0}}_{l + l' \text{ times}} + \underbrace{\mathbf{1} + \ldots + \mathbf{1}}_{j \text{ times}},$$

where $j = 1 + \sum_i (m_i + m_i') + l + l' - \natural b$. Here j is selected so that $\natural d = \natural b$.

The necessary entailments $b \Rightarrow d$ and $c \Rightarrow d$ are verified by straightforward induction on the construction of formulas b and c. ■

6 Product-free fragments of Lambek calculus

Let $\mathbf{Fm}(\backslash, /)$ stand for the set of all types in the language of two implications. In this section we show that a pair of types has a join in $\mathbf{Fm}(\backslash, /, \bullet)$ if and only if it has a join from $\mathbf{Fm}(\backslash, /)$. We also note that the diamond property for product-free fragment is different from the one in full the Lambek calculus.

6.1 Product-free joins

Theorem 8 (i) *Theorem 1 holds for $L(\backslash, /)$ and $L^*(\backslash, /)$.*

(ii) *Theorem 2 holds for $LP(\rightarrow)$ and $L^*P(\rightarrow)$.*

Here $L(\backslash, /)$ denotes the fragment of L without product. Similarly for $L^*(\backslash, /)$, $LP(\rightarrow)$ and $L^*P(\rightarrow)$.

PROOF. All the cases are similar. We give a proof for $L(\backslash, /)$ only. It suffices to prove that, if $a, b \in \mathbf{Fm}(\backslash, /)$ and $[\![a]\!] = [\![b]\!]$ in the free group, then in $\mathbf{Fm}(\backslash, /)$ there is a join for a and b. Using Theorem 1 we get a join $c \in \mathbf{Fm}(\backslash, /, \bullet)$. Lemma 13 gives a type $c' \in \mathbf{Fm}(\backslash, /)$ such that $L \vdash a \Rightarrow c'$ and $L \vdash b \Rightarrow c'$. Cut-elimination in the Lambek calculus involves that L is conservative over $L(\backslash, /)$. Hence, $a \sim b$ in $L(\backslash, /)$. ■

Lemma 13 *Let $c \in \mathbf{Fm}(\backslash, /, \bullet)$. Then:*

(i) $\exists c' \in \mathbf{Fm}(\backslash, /)$ *such that* $L \vdash c \Rightarrow c'$,

(ii) $\exists X \subset \mathbf{Fm}(\backslash, /)$ *such that* $L \vdash X \Rightarrow c$.

Proof. Induction on construction of type c.
Case $c = p$: We take $c' = p$ and $X = p$.
Case $c = a{\bullet}b$: Let $a \Rightarrow a'$, $b \Rightarrow b'$, $Y \Rightarrow a$, $Z \Rightarrow b$,
where $a', b' \in \mathbf{Fm}(\backslash, /)$ and $Y, Z \subset \mathbf{Fm}(\backslash, /)$.
We put $Z = X, Y$ and $c' = p/(b'\backslash(a'\backslash p))$ for some $p \in \mathbf{Atom}$.

$$\dfrac{\dfrac{\dfrac{b \Rightarrow b' \qquad \dfrac{a \Rightarrow a' \qquad p \Rightarrow p}{a\ ,\ a'\backslash p \Rightarrow p}(\backslash \Rightarrow)}{a\ ,\ b\ ,\ b'\backslash(a'\backslash p) \Rightarrow p}(\backslash \Rightarrow)}{a\ ,\ b \Rightarrow p/(b'\backslash(a'\backslash p))}(\Rightarrow /)}{a{\bullet}b \Rightarrow p/(b'\backslash(a'\backslash p))}(\bullet \Rightarrow) \qquad \dfrac{Y \Rightarrow a \qquad Z \Rightarrow b}{Y, Z \Rightarrow a{\bullet}b}(\Rightarrow \bullet)$$

Case $c = a\backslash b$: Let $a \Rightarrow a'$, $b \Rightarrow b'$, $\hat{a}_1, \ldots, \hat{a}_n \Rightarrow a$, $\hat{b}_1, \ldots, \hat{b}_m \Rightarrow b$
for some $a', b', \hat{a}_i, \hat{b}_j \in \mathbf{Fm}(\backslash, /)$.
We put $c' = \hat{a}_n\backslash(\ldots\backslash(\hat{a}_1\backslash b')\ldots)$ and $X = a'\backslash\hat{b}_1\ ,\ \hat{b}_2, \ldots, \hat{b}_m$.

$$\dfrac{\dfrac{\dfrac{\hat{a}_1, \ldots, \hat{a}_n \Rightarrow a \qquad b \Rightarrow b'}{\hat{a}_1, \ldots, \hat{a}_n\ ,\ a\backslash b \Rightarrow b'}(\backslash \Rightarrow)}{\vdots}(\Rightarrow \backslash)}{a\backslash b \Rightarrow \hat{a}_n\backslash(\ldots\backslash(\hat{a}_1\backslash b')\ldots)}(\Rightarrow \backslash) \qquad \dfrac{\dfrac{a \Rightarrow a' \qquad \hat{b}_1, \ldots, \hat{b}_m \Rightarrow b}{a\ ,\ a'\backslash\hat{b}_1\ ,\ \hat{b}_2, \ldots, \hat{b}_m \Rightarrow b}(\backslash \Rightarrow)}{a'\backslash\hat{b}_1\ ,\ \hat{b}_2, \ldots, \hat{b}_m \Rightarrow a\backslash b}(\Rightarrow \backslash)$$

Case $c = b/a$: Similar to the previous case. ■

6.2 Diamond property

Surprisingly, the product-free fragment of the Lambek calculus does not possess the diamond property (cf. Lemma 6). Instead of this a slightly different lemma holds.

Lemma 14 *Let $a_1, \ldots, a_n \in \mathbf{Fm}(\backslash, /)$. Then the following two assertions are equivalent.*

(i) $\exists b \in \mathbf{Fm}(\backslash, /)$ *such that* $L(\backslash, /) \vdash a_i \Rightarrow b$ *for any* $i \leq n$,

(ii) *there is a sequence X consisting of n product-free types such that*

$$L(\backslash, /) \vdash X \Rightarrow a_i \quad \textit{for any} \quad i \leq n.$$

Remark. For any n, there exist equivalent types $a_1, \ldots, a_n \in \mathbf{Fm}(\backslash, /)$ such that the sequence X in Lemma 14 cannot contain less than n types. Take for example $n = 2$, $a_1 = p/(p\backslash(p\backslash p))$ and $a_2 = q/(p\backslash(p\backslash q))$.
Obviously $L(\backslash, /) \vdash p, p \Rightarrow a_1$ and $L(\backslash, /) \vdash p, p \Rightarrow a_2$. Nevertheless, there is no product-free type d such that $L(\backslash, /) \vdash d \Rightarrow a_1$ and $L(\backslash, /) \vdash d \Rightarrow a_2$.

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