Equivalent Types in Lambek Calculus and Linear Logic

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Abstract

In 1958 J. Lambek introduced a calculus L of syntactic types and defined an equivalence relation on types: "$x \equiv y$ means that there exists a sequence $x = x_1, \ldots, x_n = y$ $(n \geq 1)$, such that $x_i \to x_{i+1}$ or $x_{i+1} \to x_i$ $(1 \leq i \leq n)$". We show that this equivalence of types is decidable for directed and non-directed Lambek calculi and for multiplicative fragments of ordinary and non-commutative linear logics. Moreover, we characterize equivalent types in these calculi in terms of simple derivability invariants (primitive type counts, balance, etc.).

1 Introduction

In [2] Joachim Lambek introduced a calculus L for deriving reduction laws of syntactic types and studied an equivalence relation on types defined as follows:
$a \sim b$ iff there is a natural number n and there exist types $c_1, \ldots, c_n$ such that $a = c_1$, $b = c_n$ and $\forall i < n$ $c_i \Rightarrow c_{i+1}$ or $c_{i+1} \Rightarrow c_i$. He also proved that this notion coincides with another one which was later called conjoinability. In [3] J. van Benthem points out that the question of decidability of this relation is still open.

In this paper we show that the equivalence of types a and b can be characterized in terms of some decidable invariants earlier introduced as necessary conditions for derivability of a sequent $a \Rightarrow b$. These invariants are primitive type counts $\#_p$ in the non-directed calculus LP and the "geometric invariant" or "balance" in the directed calculus L [3].

Since LP and L are fragments of the ordinary linear logic LL [1] and the cyclic linear logic CLL [4] respectively, we also study similar equivalence relation in these logics. Characterization of this equivalence in linear logics involves a new invariant defined by

$$\sharp p = 1, \quad \sharp(a \bullet b) = \sharp a + \sharp b - 1, \quad \sharp(a^{\perp}) = 1 - \sharp a,$$

where $\bullet$ and $(\ )^{\perp}$ denote linear conjunction and negation.

Main results of this paper are the following:

- $a \sim b$ in L iff sequent $a \Rightarrow b$ is balanced (Section 5.1).
- $a \sim b$ in LP iff $\#_p a = \#_p b$ for all literals p (Section 5.3).
- $a \sim b$ in CLL iff $\sharp a = \sharp b$ and sequent $a \Rightarrow b$ is balanced (Section 6).
- $a \sim b$ in LL iff $\sharp a = \sharp b$ and $\#_p a = \#_p b$ for all literals p (Section 6).

2 Preliminaries

2.1 Lambek calculus

In [2, p. 165] J. Lambek introduced a formal system for deriving reduction laws of syntactic types. We shall consider this system (denoted here by L^{+}) and some variants of it.

The language of Lambek calculus includes a non-empty denumerable set **Atom** of literals for primitive types and three binary connectives $\bullet$, $\backslash$, $/$, called *product* (or *fusion*), *left implication* and *right implication*. For the purpose of readability we shall omit parentheses whenever product occurres in the scope of a connective $\backslash$ or $/$. For example $p{\bullet}s/q{\bullet}r$ means $(p{\bullet}s)/(q{\bullet}r)$.

Let $p, q, r, \ldots$ stand for elements of **Atom** and letters from the beginning of alphabet $a, b, c, \ldots$ denote types built from primitive types using $\bullet$, $\backslash$ and $/$. Let capitals $X, Y, Z, \ldots$ range over finite sequences of types, possibly empty sequences. Concatenation of sequences X and Y will be denoted by X, Y.

We shall consider following Gentzen style sequent calculus L^{*}. Sequents are of the form $X \Rightarrow a$ where a is a type and X is a sequence of types. The order of types in X is essential.

The only axiom scheme is $a \Rightarrow a$, where a is any type. Rules of inference are following:

$$\frac{X \Rightarrow a \qquad Y\ ,\ b\ ,\ Z \Rightarrow c}{Y\ ,\ X\ ,\ a\backslash b\ ,\ Z \Rightarrow c}(\backslash \Rightarrow) \qquad\qquad \frac{a, X \Rightarrow b}{X \Rightarrow a\backslash b}(\Rightarrow \backslash)$$

$$\frac{X \Rightarrow a \qquad Y\ ,\ b\ ,\ Z \Rightarrow c}{Y\ ,\ b/a\ ,\ X\ ,\ Z \Rightarrow c}(/ \Rightarrow) \qquad\qquad \frac{X, a \Rightarrow b}{X \Rightarrow b/a}(\Rightarrow /)$$

$$\frac{X\ ,\ a\ ,\ b\ ,\ Y \Rightarrow c}{X\ ,\ a{\bullet}b\ ,\ Y \Rightarrow c}(\bullet\Rightarrow) \qquad\qquad \frac{X \Rightarrow a \quad Y \Rightarrow b}{X, Y \Rightarrow a{\bullet}b}(\Rightarrow\bullet)$$

$$\frac{X \Rightarrow a \quad Y, a, Z \Rightarrow b}{Y, X, Z \Rightarrow b}(\text{CUT})$$

We write $L^* \vdash X \Rightarrow a$ for "sequent $X \Rightarrow a$ is derivable in L^* ".

Example 1 $L^* \vdash p, (p\backslash p)/p, p \Rightarrow p$

$$\frac{p \Rightarrow p \qquad \dfrac{p \Rightarrow p \quad p \Rightarrow p}{p\ ,\ p\backslash p \Rightarrow p}(\backslash\Rightarrow)}{p\ ,\ (p\backslash p)/p\ ,\ p \Rightarrow p}(/\Rightarrow)$$

Given a formal system T, we shall write $a \overset{T}{\cong} b$ iff $T \vdash a \Rightarrow b$ and $T \vdash b \Rightarrow a$.

Example 2 $b/a{\bullet}c \overset{L^*}{\cong} (b/c)/a$

It turns out that if $a \overset{T}{\cong} b$ then replacing an instance of type a in a sequent by type b does not have any effect on derivability in T. In particular $(a{\bullet}b){\bullet}c \overset{L^+}{\cong} a{\bullet}(b{\bullet}c)$ and $(a\backslash b)/c \overset{L^+}{\cong} a\backslash(b/c)$ for any types a, b and c. So, we can omit parentheses in types $(a{\bullet}b){\bullet}c$ and $(a\backslash b)/c$.

Adding to L^* the permutation rule

$$\frac{X, a, b, Y \Rightarrow c}{X, b, a, Y \Rightarrow c}(\text{P})$$

we obtain the non-directed Lambek calculus L^*P. In L^*P we have $a\backslash b \overset{L^*P}{\cong} b/a$ for any types a and b, whence left and right implications collaps into one connective which is often denoted by $a{\rightarrow}b$.

Calculi L^+ and L^+P (usually denoted by L and LP) are obtained from L^* and L^*P respectively, by adding the constraint that all sequents occurring in a derivation must have non-empty antecedents. In other words, in L^+ and L^+P rules $(\Rightarrow\backslash)$ and $(\Rightarrow/)$ may be applied only if the sequence X is not empty.

This requirement excludes sequents like $L^* \vdash (p\backslash p)\backslash q \Rightarrow q$, which are non-adequate from the linguistic point of view.

Evidently following inclusions hold:

$$\begin{array}{ccc} L^+ & \subset & L^+P \\ \cap & & \cap \\ L^* & \subset & L^*P \end{array}$$

We define now a notion of duality which often allows to cut by half proofs about derivability in Lambek calculi.

$$\begin{array}{rcl} \text{dual }(p) & \rightleftharpoons & p \\ \text{dual }(a{\bullet}b) & \rightleftharpoons & \text{dual }(b){\bullet}\text{ dual }(a) \\ \text{dual }(a\backslash b) & \rightleftharpoons & \text{dual }(b)/\text{ dual }(a) \\ \text{dual }(a/b) & \rightleftharpoons & \text{dual }(b)\backslash\text{ dual }(a) \\ \text{dual }(a_1,\ldots,a_n \Rightarrow b) & \rightleftharpoons & \text{dual }(a_n),\ldots,\text{ dual }(a_1) \Rightarrow \text{ dual }(b) \end{array}$$

In any of the calculi considered here derivability of a sequent is equivalent to derivability of its dual.

2.2 Linear logic

We shall consider mainly the multiplicative fragment of linear logic, although the conservativity results hold also for full linear logic.

First we introduce LL^* — the linear logic counterpart of L^*P. The only connectives of LL^* are linear negation, linear conjunction and linear disjunction. We denote them by $(\ )^\perp$, $\bullet$ and $+$ respectively.

The formulas of linear logic (called *linear formulas* for shortness) are defined as follows:

- if $p \in \mathbf{Atom}$ then p and $p^\perp$ are formulas,
- if a and b are formulas then $a{\bullet}b$ and $a{+}b$ are also formulas.

Negation of a formula is defined thus:

$$\begin{array}{rcl} (p)^\perp & \rightleftharpoons & p^\perp \\ (p^\perp)^\perp & \rightleftharpoons & p \\ (a{\bullet}b)^\perp & \rightleftharpoons & b^\perp{+}a^\perp \\ (a{+}b)^\perp & \rightleftharpoons & b^\perp{\bullet}a^\perp \end{array}$$

The order of $b^\perp$ and $a^\perp$ in definition of $(a{\bullet}b)^\perp$ and $(a{+}b)^\perp$ is not important in ordinary linear logic, but it becomes essential in non-commutative linear logic.

Derivable objects of linear logic are sequents $\Rightarrow a_1;\ldots;a_n$, where a_1 , … , a_n are formulas.

We shall interpret $\Rightarrow a_1;\ldots;a_n$ as $a_1{+}\cdots{+}a_n$.

The axiom scheme is $\Rightarrow a^\perp; a$. Rules of inference are following:

$$\frac{\Rightarrow X\ ;\ a\ ;\ b\ ;\ Y}{\Rightarrow X\ ;\ a{+}b\ ;\ Y}(+) \qquad\qquad \frac{\Rightarrow X; a \qquad \Rightarrow b; Y}{\Rightarrow X\ ;\ a{\bullet}b\ ;\ Y}(\bullet)$$

$$\frac{\Rightarrow X;a \qquad \Rightarrow a^{\perp}\ ;\ Y}{\Rightarrow X;Y}\text{(CUT)} \qquad\qquad \frac{\Rightarrow X;a;b;Y}{\Rightarrow X;b;a;Y}\text{(P)}$$

The cyclic (non-commutative) linear logic CLL^* is obtained from LL^* by replacing the permutation rule (P) with the rotation rule

$$\frac{\Rightarrow X;Y}{\Rightarrow Y;X}\text{(ROT)}$$

The calculus CLL^* defined here is a $(\bullet, +, (\ )^{\perp})$-fragment of cyclic linear logic CLL presented by Yetter in [4].

If we demand that every sequent in a derivation should contain at least two formulas, we obtain calculi LL^+ and CLL^+ which are the linear logic counterparts of L^+P and L^+ respectively.

Obviously following inclusions hold:

$$\begin{array}{ccc} CLL^+ & \subset & LL^+ \\ \cap & & \cap \\ CLL^* & \subset & LL^* \end{array}$$

It is convenient to have in linear logic a notion of entailment, similar to $a \Rightarrow b$ in Lambek calculus. We shall write $LL^* \vdash a \Rightarrow b$ iff $LL^* \vdash\ \Rightarrow a^{\perp}; b$. Similar denotation will also be used in other linear calculi.

Again formulas a and b equivalent in the sense of $a \cong b$ in corresponding logic, may be replaced by each other. Immediate verification shows that $a \bullet (b \bullet c) \overset{LL^+}{\cong} (a \bullet b) \bullet c$ and $a + (b + c) \overset{LL^+}{\cong} (a + b) + c$. Hence we may omit parentheses in these expressions. We shall also omit parentheses in $a + (b \bullet c)$, but not in $(a + b) \bullet c$.

The phenomenon of duality is present in linear logic as well.

$$\begin{array}{rcl} \text{dual}\ (p) & \rightleftharpoons & p \\ \text{dual}\ (p^{\perp}) & \rightleftharpoons & p^{\perp} \\ \text{dual}\ (a \bullet b) & \rightleftharpoons & \text{dual}\ (b) \bullet \text{dual}\ (a) \\ \text{dual}\ (a + b) & \rightleftharpoons & \text{dual}\ (b) + \text{dual}\ (a) \\ \text{dual}\ (\Rightarrow a_1\ ;\ \ldots\ ;\ a_n) & \rightleftharpoons & \Rightarrow\ \text{dual}\ (a_n)\ ;\ \ldots\ ;\ \text{dual}\ (a_1) \end{array}$$

Cut-elimination holds in all the calculi considered in this paper (cf. [2] [3] [4]).

3 Soundness results

In this section we extend the definitions of balance and primitive type counts in Lambek calculus (cf. [3]) to the case of linear logic. We present balance in group-theoretic terms.

A numerical model called *sharp* is introduced. Both ordinary and cyclic linear logic are sound with respect to this additional invariant.

All the lemmas in this section are proved by straightforward induction. We omit the proofs of most of them.

3.1 Balance

Let Γ be the set of all symbols for primitive types $p, q, \ldots$ and their opposites $p^{-1}, q^{-1}, \ldots$, i.e. $\Gamma \rightleftharpoons \mathbf{Atom} \cup (\mathbf{Atom})^{-1}$. Then Γ^* denotes the set of all words in the alphabet Γ. We shall use $\alpha, \beta, \gamma, \ldots$ for words from Γ^*. Let Λ denote the empty word.

The operation *opposite* $(\ )^{-1}$ on Γ^* is defined in the natural way.

$$\begin{array}{rcl} (p)^{-1} & \rightleftharpoons & p^{-1} \\ (p^{-1})^{-1} & \rightleftharpoons & p \\ (\alpha\beta)^{-1} & \rightleftharpoons & (\beta)^{-1}(\alpha)^{-1} \end{array}$$

To each type a in the language of Lambek calculus corresponds an *atomic marking* $[[a]] \in \Gamma^* \setminus \{\Lambda\}$ defined as follows :

$$\begin{array}{rcl} [[p]] & \rightleftharpoons & p \\ [[a \bullet b]] & \rightleftharpoons & [[a]][[b]] \\ [[a \backslash b]] & \rightleftharpoons & ([[a]])^{-1}[[b]] \\ [[b/a]] & \rightleftharpoons & [[b]]([[a]])^{-1} \end{array}$$

Let $\alpha \equiv \beta$ stand for graphical identity of words α and β.

We shall now interpret concatenation and $(\ )^{-1}$ as group operations.

Let $\alpha = \beta$ stand for equality of elements of Γ^* in the free group generated by primitive types $p, q, r, \ldots$. In other words "=" denotes the reflexive symmetric transitive closure of identities

$$\delta p p^{-1} \epsilon = \delta\epsilon \quad \text{and} \quad \delta p^{-1} p \epsilon = \delta\epsilon \quad \text{for any} \quad \delta, \epsilon \in \Gamma^*, p \in \mathbf{Atom} \ .$$

Definition. A sequent $a_1, \ldots, a_n \Rightarrow b$ is *balanced* iff $[[a_1]] \cdots [[a_n]] = [[b]]$ in the free group generated by **Atom** .

It is easy to see that this definition of balance is equivalent to the one given in [3].

Example 3 The sequent $p/q, q \Rightarrow p$ is balanced, since $[[(p/q) \bullet q]] \equiv pq^{-1}q$, $[[p]] \equiv p$ and $pq^{-1}q = p$ in the free group.

Lemma 1 (cf. [3]) *If $L^* \vdash X \Rightarrow a$ then $X \Rightarrow a$ is balanced.*

Proof. Induction on derivations. We omit the trivial cases of axioms and the rule $(\bullet \Rightarrow\ )$.
Case $(\ \Rightarrow \bullet)$: If $[[X]] = [[a]]$ and $[[Y]] = [[b]]$ then obviously $[[X]][[Y]] = [[a]][[b]]$.
Case $(\ \Rightarrow \backslash)$: Multiplying the equality $[[a]][[X]] = [[b]]$ by $[[a]]^{-1}$ on the left, one obtains $[[X]] = [[a]]^{-1}[[b]]$ as desired.
Case $(\backslash \Rightarrow\ )$: If $[[X]] = [[a]]$ then evidently $[[X]][[a]]^{-1} = \Lambda$. Now $[[Y]][[b]][[Z]] = [[c]]$ entails $[[Y]][[X]][[a]]^{-1}[[b]][[Z]] = [[c]]$.
Case $(\ \Rightarrow /)$ and $(/ \Rightarrow\ )$: The dual rules are treated similarly. ■

In linear logic atomic markings are defined by:

$$
\begin{array}{rcl}
[[p]] & \rightleftharpoons & p \\
[[p^{\perp}]] & \rightleftharpoons & p^{-1} \\
[[a \bullet b]] & \rightleftharpoons & [[a]][[b]] \\
[[a + b]] & \rightleftharpoons & [[a]][[b]]
\end{array}
$$

Definition. A sequent of cyclic linear logic $\Rightarrow a_1\ ;\ \ldots\ ;\ a_n$ is *balanced* iff $[[a_1]] \cdots [[a_n]] = \Lambda$ in the free group generated by **Atom** .

Lemma 2 (i) *If* $CLL^* \vdash\ \Rightarrow X$ *then* $\Rightarrow X$ *is balanced.*

(ii) *In particular, if* $CLL^* \vdash a \Rightarrow b$ *then* $[[a]] = [[b]]$ *in the free group.*

3.2 Primitive type counts

Definition. For any primitive type $p \in$ **Atom**, *p-count* $\#_p$ is the following mapping from types to integers.

$$
\begin{array}{rcl}
\#_p p & \rightleftharpoons & 1 \\
\#_p q & \rightleftharpoons & 0 \quad \text{, if } p \text{ and } q \text{ are distinct primitive types} \\
\#_p(a \bullet b) & \rightleftharpoons & \#_p a + \#_p b \\
\#_p(a \backslash b) & \rightleftharpoons & \#_p b - \#_p a \\
\#_p(b / a) & \rightleftharpoons & \#_p b - \#_p a
\end{array}
$$

Lemma 3 (cf. [3]) *If* $L^*P \vdash a_1, \ldots, a_n \Rightarrow b$
then $\#_p a_1 + \cdots + \#_p a_n = \#_p b$ *for any* $p \in$ **Atom** .

Definition. For linear formulas we define *p-count* as follows:

$$\begin{array}{rcl}
\#_p p & \rightleftharpoons & 1 \\
\#_p(p^\perp) & \rightleftharpoons & -1 \\
\#_p q & \rightleftharpoons & 0 \ \text{, if } p \text{ and } q \text{ are distinct atomic formulas} \\
\#_p(q^\perp) & \rightleftharpoons & 0 \ \text{, if } p \text{ and } q \text{ are distinct atomic formulas} \\
\#_p(a{\bullet}b) & \rightleftharpoons & \#_p a + \#_p b \\
\#_p(a{+}b) & \rightleftharpoons & \#_p a + \#_p b
\end{array}$$

Lemma 4 (i) *If* $LL^* \vdash \ \Rightarrow a_1 \ ; \ \ldots \ ; \ a_n$ *then* $\#_p a_1 + \cdots + \#_p a_n = 0$ *for any* $p \in \mathbf{Atom}$ *.*

(ii) *In particular, if* $LL^* \vdash a \Rightarrow b$ *then* $\#_p a = \#_p b$ *for any* $p \in \mathbf{Atom}$ *.*

3.3 Sharp

Definition. *Sharp* is the following mapping $\sharp$ from types of linear logic to integers.

$$\begin{array}{rcl}
\sharp p & \rightleftharpoons & 1 \\
\sharp p^\perp & \rightleftharpoons & 0 \\
\sharp(a{+}b) & \rightleftharpoons & \sharp a + \sharp b \\
\sharp(a{\bullet}b) & \rightleftharpoons & \sharp a + \sharp b - 1
\end{array}$$

Lemma 5 $\sharp(a^\perp) = 1 - \sharp a$ *for any linear formula* a*.*

Lemma 6 (i) *If* $LL^* \vdash \ \Rightarrow a_1 \ ; \ \ldots \ ; \ a_n$ *then* $\sharp a_1 + \cdots + \sharp a_n = 0$*.*

(ii) *In particular, if* $LL^* \vdash a \Rightarrow b$ *then* $\sharp a = \sharp b$*.*

We note that Lemma 1 can also be obtained as a corollary of Lemma 2 and Theorem 5. Similarly, Lemma 3 follows from Lemma 4 and Theorem 5.

4 Equivalent types

All the results of this section hold in any of the eight systems introduced in preliminaries. For simplicity, we shall prove them only for L^+.

Definition. Given a formal system T and two types a and b in the language of T, we write $a \overset{T}{\leq} b$ iff sequent $a \Rightarrow b$ is derivable in theory T. Let $\overset{T}{\sim}$ stand for the reflexive, symmetric and transitive closure of relation $\overset{T}{\leq}$. We say that two types a and b are *equivalent* in theory T iff $a \overset{T}{\sim} b$. We shall write $a \sim b$ instead of $a \overset{L^+}{\sim} b$, since derivability in L^* involves derivability in any of the calculi introduced in preliminaries.

Remark. If $T \vdash a \Rightarrow b$ or $T \vdash b \Rightarrow a$ then, obviously, $a \overset{T}{\sim} b$. However, the following example shows that these cases do not exhaust the notion of equivalence.

Example 4 $b/(a\backslash a/a) \overset{L^+}{\sim} b{\bullet}a$ for any types a and b.

The following derivations show that $b{\bullet}a/(a\backslash a/a){\bullet}a$ is a consequent of both $b/(a\backslash a/a)$ and $b{\bullet}a$ in L^+.

$$\dfrac{\dfrac{\dfrac{\dfrac{a\backslash a/a \Rightarrow a\backslash a/a \qquad b \Rightarrow b}{b/(a\backslash a/a)\ ,\ a\backslash a/a \Rightarrow b}(/\Rightarrow) \qquad a \Rightarrow a}{b/(a\backslash a/a)\ ,\ a\backslash a/a\ ,\ a \Rightarrow b{\bullet}a}(\Rightarrow \bullet)}{b/(a\backslash a/a)\ ,\ (a\backslash a/a){\bullet}a \Rightarrow b{\bullet}a}(\bullet \Rightarrow)}{b/(a\backslash a/a) \Rightarrow b{\bullet}a/(a\backslash a/a){\bullet}a}(\Rightarrow /)$$

$$\dfrac{\dfrac{\dfrac{b \Rightarrow b \qquad a\ ,\ a\backslash a/a\ ,\ a \Rightarrow a}{b\ ,\ a\ ,\ a\backslash a/a\ ,\ a \Rightarrow b{\bullet}a}(\Rightarrow \bullet)}{b{\bullet}a\ ,\ (a\backslash a/a){\bullet}a \Rightarrow b{\bullet}a}(\bullet \Rightarrow)}{b{\bullet}a \Rightarrow b{\bullet}a/(a\backslash a/a){\bullet}a}(\Rightarrow /)$$

Next two lemmas belonging to J. Lambek [2] show that equivalence of types coincides with conjoinability defined in [3].

Lemma 7 (Diamond property.) *Let a and b be any types of L. Then following two assertions are equivalent.*

(i) *There exists a type c such that $L \vdash a \Rightarrow c$ and $L \vdash b \Rightarrow c$.*

(ii) *There exists a type d such that $L \vdash d \Rightarrow a$ and $L \vdash d \Rightarrow b$.*

$$\begin{tikzcd}
 & a \arrow[dr] & \\
d \arrow[ur] \arrow[dr] & & c \\
 & b \arrow[ur] &
\end{tikzcd}$$

In other words, we can find any of the types c or d indicated on the figure, if other three types are given.

Proof. We give a proof slightly shorter than in [2].

Case (i)→(ii): We put $d = (a/c)\bullet c\bullet(c\backslash b)$.

$$\cfrac{\cfrac{c \Rightarrow c \qquad \cfrac{b \Rightarrow c \quad a \Rightarrow a}{a/c\ ,\ b \Rightarrow a}(/\Rightarrow)}{a/c\ ,\ c\ ,\ c\backslash b \Rightarrow a}(\backslash\Rightarrow)}{(a/c)\bullet c\bullet(c\backslash b) \Rightarrow a}(\bullet\Rightarrow)$$

The sequent $(a/c)\bullet c\bullet(c\backslash b) \Rightarrow b$ is derived dually.
Case (ii)→(i): We put $c = (d/a)\backslash d/(b\backslash d)$.

$$\cfrac{\cfrac{\cfrac{a \Rightarrow a \qquad \cfrac{d \Rightarrow b \quad d \Rightarrow d}{d\ ,\ b\backslash d \Rightarrow d}(\backslash\Rightarrow)}{d/a\ ,\ a\ ,\ b\backslash d \Rightarrow d}(/\Rightarrow)}{a\ ,\ b\backslash d \Rightarrow (d/a)\backslash d}(\Rightarrow\backslash)}{a \Rightarrow (d/a)\backslash d/(b\backslash d)}(\Rightarrow/)$$

The sequent $b \Rightarrow (d/a)\backslash d/(b\backslash d)$ is derived similarly. ■

Lemma 8 *Two types a and b are equivalent if and only if the assertions (i) and (ii) from the previous lemma hold.*

Proof. 'If' part is obvious. Two prove 'only if' part, we assume that there are n types $e_1, \ldots, e_n$ such that $\forall i < n\ \ e_i \Rightarrow e_{i+1}$ or $e_{i+1} \Rightarrow e_i$. Now (i) of Lemma 7 follows from (ii)→(i) and

$$\frac{e_i \Rightarrow e_{i+1} \qquad e_{i+1} \Rightarrow e_{i+2}}{e_i \Rightarrow e_{i+2}}(\mathrm{CUT})$$

by induction on n. ■
Definition. We say that type c is a *join* for a set of types $\{a_1, \ldots, a_n\}$ iff $\forall i \leq n\ \ L^+ \vdash a_i \Rightarrow c$.

Corollary 9 *A finite set of types has a join if and only if the types are pairwise conjoinable.*

Lemma 10 (Replacement property of equivalence.) *Let c_a be a type containing a type a as a specified part, and let c_b come from c_a by replacing that part by a type b. If $a \sim b$, then $c_a \sim c_b$.*

Proof. Induction on the construction of type c_a. It suffices to prove three facts.

(i) If $a_1 \sim b_1$ and $a_2 \sim b_2$ then $a_1\bullet a_2 \sim b_1\bullet b_2$.

(ii) If $a_1 \sim b_1$ and $a_2 \sim b_2$ then $a_1\backslash a_2 \sim b_1\backslash b_2$.

(iii) If $a_1 \sim b_1$ and $a_2 \sim b_2$ then $a_1/a_2 \sim b_1/b_2$.

Case (i): Let $L \vdash a_1 \Rightarrow c_1$, $L \vdash b_1 \Rightarrow c_1$ and $L \vdash a_2 \Rightarrow c_2$, $L \vdash b_2 \Rightarrow c_2$.

$$\dfrac{\dfrac{a_1 \Rightarrow c_1 \qquad a_2 \Rightarrow c_2}{a_1, a_2 \Rightarrow c_1 \bullet c_2}(\Rightarrow \bullet)}{a_1 \bullet a_2 \Rightarrow c_1 \bullet c_2}(\bullet \Rightarrow) \qquad \dfrac{\dfrac{b_1 \Rightarrow c_1 \qquad b_2 \Rightarrow c_2}{b_1, b_2 \Rightarrow c_1 \bullet c_2}(\Rightarrow \bullet)}{b_1 \bullet b_2 \Rightarrow c_1 \bullet c_2}(\bullet \Rightarrow)$$

Case (ii): Let $L \vdash a_1 \Rightarrow c_1$, $L \vdash b_1 \Rightarrow c_1$ and $L \vdash a_2 \Rightarrow c_2$, $L \vdash b_2 \Rightarrow c_2$.
According to diamond property there exists a type d_1 such that $L \vdash d_1 \Rightarrow a_1$ and $L \vdash d_1 \Rightarrow b_1$.

$$\dfrac{\dfrac{d_1 \Rightarrow a_1 \qquad a_2 \Rightarrow d_2}{d_1\ ,\ a_1\backslash a_2 \Rightarrow d_2}(\backslash \Rightarrow)}{a_1\backslash a_2 \Rightarrow d_1\backslash d_2}(\Rightarrow \backslash) \qquad \dfrac{\dfrac{d_1 \Rightarrow b_1 \qquad b_2 \Rightarrow d_2}{d_1\ ,\ b_1\backslash b_2 \Rightarrow d_2}(\backslash \Rightarrow)}{b_1\backslash b_2 \Rightarrow d_1\backslash d_2}(\Rightarrow \backslash)$$

Case (iii): This case is the dual of the previous one. ■

5 Equivalent types in Lambek calculus

In this section we prove that the notion of equivalence of two types in the directed Lambek calculus coincides with the balance and in the undirected Lambek calculus it coincides with the equality of primitive type counts.

5.1 Pure Lambek calculus

Theorem 1 *For any types a and b the following three clauses are equivalent.*

(i) $a \overset{L^+}{\sim} b$

(ii) $a \overset{L^*}{\sim} b$

(iii) $[[a]] = [[b]]$ *in the free group.*

The implication (i)→(ii) follows from $L^+ \subset L^*$.
Proof of (ii)→(iii) . Let $L^* \vdash a \Rightarrow c$ and $L^* \vdash b \Rightarrow c$. By Lemma 1 this entails $[[a]] = [[c]]$ and $[[b]] = [[c]]$. Hence $[[a]] = [[b]]$. ■

Before proving (iii)→(i) we introduce the notion of simple products.

Let **SP** stand for the set of all products of factors of the form p or $(p\backslash p/p)$, where $p \in$ **Atom**. We shall call these types *simple products*.

There is a natural mapping $\mathbf{sp}(\ )$ from atomic markings $\Gamma^* \setminus \{\Lambda\}$ to simple products $\mathbf{SP}$.

$$\begin{array}{rcl} \mathbf{sp}(p) & \rightleftharpoons & p \\ \mathbf{sp}(p^{-1}) & \rightleftharpoons & p\backslash p/p \\ \mathbf{sp}(\alpha\beta) & \rightleftharpoons & \mathbf{sp}(\alpha){\bullet}\mathbf{sp}(\beta) \end{array}$$

Now we deduce (iii)$\to$(i) from following auxiliary results which will be proved in Section 5.2.

- Every type is equivalent to a simple product (cf. Lemma 12).
- Simple products corresponding to equal (in the free group) atomic markings are equivalent (cf. Lemma 13).

Proof of Theorem 1 (iii)$\to$(i) . If $[[a]] = [[b]]$ then by Lemma 13 we have

$$\mathbf{sp}([[a]]) \overset{L^+}{\sim} \mathbf{sp}([[b]]).$$

Lemma 12 proves that $a \overset{L^+}{\sim} \mathbf{sp}([[a]])$ and $b \overset{L^+}{\sim} \mathbf{sp}([[b]])$. Now $a \overset{L^+}{\sim} b$ follows by transitivity. ■

5.2 Simple products

Following is a technical lemma, used in the proof of Lemma 12.

Lemma 11 *For any atomic marking $\alpha \in \Gamma^* \setminus \{\Lambda\}$ and any type b*

(i) $b/\mathbf{sp}(\alpha) \sim b{\bullet}\mathbf{sp}(\alpha^{-1})$

(ii) $\mathbf{sp}(\alpha)\backslash b \sim \mathbf{sp}(\alpha^{-1}){\bullet}b$

Proof of (i). Let $\alpha = \sigma_1 \ldots \sigma_n$ where $\sigma_1, \ldots, \sigma_n \in \Gamma$. Then type $b/\mathbf{sp}(\alpha)$ is the same as $b/\mathbf{sp}(\sigma_1){\bullet}\cdots{\bullet}\mathbf{sp}(\sigma_n)$. By Example 2 we have

$$b/\mathbf{sp}(\sigma_1){\bullet}\cdots{\bullet}\mathbf{sp}(\sigma_n) \cong (\ldots(b/\mathbf{sp}(\sigma_n))/\ldots)/\mathbf{sp}(\sigma_1).$$

On the other hand, type $b{\bullet}\mathbf{sp}(\alpha^{-1})$ is the same as $b{\bullet}\mathbf{sp}(\sigma_1^{-1}){\bullet}\cdots{\bullet}\mathbf{sp}(\sigma_n^{-1})$.

It remains to consider the case $\alpha = \sigma_1$.

Case $\alpha = p$: We prove that $b/p \sim b{\bullet}(p\backslash p/p)$.

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p \qquad \dfrac{b \Rightarrow b \quad p \Rightarrow p}{p/b\ ,\ b \Rightarrow p}(/\Rightarrow)}{p/b\ ,\ b/p\ ,\ p \Rightarrow p}(/\Rightarrow)}{p/b\ ,\ b/p \Rightarrow p/p}(\Rightarrow /)}{b/p \Rightarrow (p/b)\backslash p/p}(\Rightarrow \backslash)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{b \Rightarrow b \qquad p\ ,\ p\backslash p/p\ ,\ p \Rightarrow p}{p/b\ ,\ b\ ,\ p\backslash p/p\ ,\ p \Rightarrow p}(/\Rightarrow)}{p/b\ ,\ b\ ,\ p\backslash p/p \Rightarrow p/p}(\Rightarrow /)}{b\ ,\ p\backslash p/p \Rightarrow (p/b)\backslash p/p}(\Rightarrow \backslash)}{b{\bullet}(p\backslash p/p) \Rightarrow (p/b)\backslash p/p}({\bullet}\Rightarrow)$$

Case $\alpha = p^{-1}$: According to Example 4 $b/(p\backslash p/p) \sim b{\bullet}p$. $\blacksquare$

The clause (ii) is proved similarly.

Lemma 12 $a \sim \mathbf{sp}([[a]])$ *for any type* a

Proof. Induction on the construction of type a.
Case $a = p$: Obviously, $p \sim p$.
Case $a = b{\bullet}c$: Follows immediately from replacement property (cf. Lemma 10 (i)).
Case $a = b/c$: According to definitions of $\mathbf{sp}(\ )$ and $[[\ ]]$, it is sufficient to prove $b/c \sim \mathbf{sp}([[b]]){\bullet}\mathbf{sp}([[c]]^{-1})$. By induction hypothesis and replacement property we have $b/c \sim \mathbf{sp}([[b]])/\mathbf{sp}([[c]])$. Using Lemma 11 (i) we get $\mathbf{sp}([[b]])/\mathbf{sp}([[c]]) \sim \mathbf{sp}([[b]]){\bullet}\mathbf{sp}([[c]]^{-1})$.

Case $a = b\backslash c$: This case is treated symmetrically to the previous one using Lemma 11 (ii).
$\blacksquare$

Lemma 13 *Let two atomic markings* α *and* β *be equal in the free group. Then* $\mathbf{sp}(\alpha) \sim \mathbf{sp}(\beta)$.

Proof. Equality of α and β in the free group means that there is a natural number n and there exist words $\gamma_i \in \Gamma^*$, such that $\alpha \equiv \gamma_1$, $\beta \equiv \gamma_n$ and $\gamma_1 = \gamma_2 = \ldots = \gamma_n$ where each equality $\gamma_i = \gamma_{i+1}$ is of the form $\delta p p^{-1} \epsilon = \delta\epsilon$ or $\delta p^{-1} p \epsilon = \delta\epsilon$ for some $\delta, \epsilon \in \Gamma^*$ and $p \in \mathbf{Atom}$. Replacing fragment $p^{-1}p = \Lambda = q^{-1}q$ with $p^{-1}p = p^{-1}pq^{-1}q = q^{-1}q$, we can choose all γ_i different from the empty word Λ.

In order to prove $\mathbf{sp}(\alpha) \sim \mathbf{sp}(\beta)$, it suffices to show that $\mathbf{sp}(\gamma_i) \sim \mathbf{sp}(\gamma_{i+1})$ for all $i < n$. We break the case $\mathbf{sp}(\delta p p^{-1}\epsilon) \sim \mathbf{sp}(\delta\epsilon)$ down to three subcases, depending on whether ϵ or δ or both are empty words. We omit the proof of the dual law $\mathbf{sp}(\delta p^{-1} p\epsilon) \sim \mathbf{sp}(\delta\epsilon)$.
Case (i): $d{\bullet}p{\bullet}(p\backslash p/p) \sim d$
As a join we take $d{\bullet}p/p$.

$$\dfrac{\dfrac{\dfrac{d \Rightarrow d \qquad p\ ,\ p\backslash p/p\ ,\ p \Rightarrow p}{d\ ,\ p\ ,\ p\backslash p/p\ ,\ p \Rightarrow d{\bullet}p}{\scriptstyle(\Rightarrow\bullet)}}{d{\bullet}p{\bullet}(p\backslash p/p)\ ,\ p \Rightarrow d{\bullet}p}{\scriptstyle(\bullet\Rightarrow)}}{d{\bullet}p{\bullet}(p\backslash p/p) \Rightarrow d{\bullet}p/p}{\scriptstyle(\Rightarrow /)} \qquad\qquad \dfrac{\dfrac{d \Rightarrow d \qquad p \Rightarrow p}{d\ ,\ p \Rightarrow d{\bullet}p}{\scriptstyle(\Rightarrow\bullet)}}{d \Rightarrow d{\bullet}p/p}{\scriptstyle(\Rightarrow /)}$$

Case (ii): $p{\bullet}(p\backslash p/p){\bullet}e \sim e$
Here $p/(e\backslash p)$ is a join.

$$\dfrac{\dfrac{\dfrac{e \Rightarrow e \qquad p\ ,\ p\backslash p/p\ ,\ p \Rightarrow p}{p\ ,\ p\backslash p/p\ ,\ e\ ,\ e\backslash p \Rightarrow p}{\scriptstyle(\backslash\Rightarrow)}}{p{\bullet}(p\backslash p/p){\bullet}e\ ,\ e\backslash p \Rightarrow p}{\scriptstyle(\bullet\Rightarrow)}}{p{\bullet}(p\backslash p/p){\bullet}e \Rightarrow p/(e\backslash p)}{\scriptstyle(\Rightarrow /)} \qquad\qquad \dfrac{\dfrac{e \Rightarrow e \qquad p \Rightarrow p}{e\ ,\ e\backslash p \Rightarrow p}{\scriptstyle(\backslash\Rightarrow)}}{e \Rightarrow p/(e\backslash p)}{\scriptstyle(\Rightarrow /)}$$

Case (iii): $d{\bullet}p{\bullet}(p\backslash p/p){\bullet}e \sim d{\bullet}e$
Follows immediately from any of (i) or (ii) by replacement property (cf. Lemma 10 (i)). This completes the proof of Lemma 13. $\blacksquare$

5.3 Lambek calculus with permutation

Theorem 2 *For any types a and b the following three clauses are equivalent.*

(i) $a \overset{L^+P}{\sim} b$

(ii) $a \overset{L^*P}{\sim} b$

(iii) $\#_p a = \#_p b$ *for any* $p \in \mathbf{Atom}$.

Again (i)→(ii) is obvious and (ii)→(iii) follows from soundness of L^*P with respect to primitive type counts (cf. Lemma 3).

To prove (iii)→(i) we proceed like in Theorem 1. Evidently Lemma 8 and Lemma 12 hold in L^+P as well. In order to prove an analog of Lemma 13, we note that "$\#_p a = \#_p b$ for any $p \in \mathbf{Atom}$" means that $[[a]]$ and $[[b]]$ are equal in the free Abelian group generated by **Atom** .

Lemma 14 *If two atomic markings α and β are equal in the free Abelian group then* $\mathbf{sp}(\alpha) \overset{L^+P}{\sim} \mathbf{sp}(\beta)$.

Proof. Equality in this group may be considered as the transitive closure of $\delta p p^{-1} \epsilon = \delta \epsilon$ and $\delta \alpha \beta \epsilon = \delta \beta \alpha \epsilon$. To modify the proof of Lemma 13 it is sufficient to show that

$$d\bullet a\bullet b\bullet e \overset{L^+P}{\sim} d\bullet b\bullet a\bullet e.$$

Actually even $d\bullet a\bullet b\bullet e \overset{L^+P}{\cong} d\bullet b\bullet a\bullet e$.

$$\dfrac{\dfrac{\dfrac{\dfrac{d \Rightarrow d \quad b \Rightarrow b}{d, b \Rightarrow d\bullet b}(\Rightarrow\bullet) \qquad \dfrac{a \Rightarrow a \quad e \Rightarrow e}{a, e \Rightarrow a\bullet e}(\Rightarrow\bullet)}{d, b, a, e \Rightarrow d\bullet b\bullet a\bullet e}(\Rightarrow\bullet)}{d, a, b, e \Rightarrow d\bullet b\bullet a\bullet e}(\mathrm{P})}{d\bullet a\bullet b\bullet e \Rightarrow d\bullet b\bullet a\bullet e}(\bullet\Rightarrow)$$

This completes the proof of Lemma 14. ■

6 Eqivalent types in linear logic

In this section we give a simple criterion for the equivalence of types in rudimentary fragments of linear logic and cyclic linear logic.

In full linear logic any two formulas a and b are trivially equivalent by

$$a \Rightarrow a \oplus b \quad \text{and} \quad b \Rightarrow a \oplus b \quad ,$$

where $a \oplus b$ stands for non-linear (additive) disjunction, sometimes also denoted by $a \sqcup b$. In multiplicative fragment the situation is different.

By LL^* we denote the fragment of ordinary linear logic without exponentials, additives and units. Similarly, CLL^* stands for corresponding fragment of cyclic (non-commutative) linear logic. By LL^+ and CLL^+ we denote the subtheories, where single-formula sequents are not allowed.

Theorem 3 *Let a and b be any two formulas in the language $\bullet$, $+$, $(\ )^\perp$. Then following three clauses are equivalent.*

(i) $a \overset{CLL^+}{\sim} b$

(ii) $a \overset{CLL^*}{\sim} b$

(iii) $\sharp a = \sharp b$ *and* $[[a]] = [[b]]$ *in the free group generated by* **Atom** .

Implication (i)→(ii) is obvious. Implication (ii)→(iii) is an easy consequence of soundness with respect to $\sharp$ and balance (cf. Lemma 6 (ii) and Lemma 2 (ii)).

Before proving (iii)→(i) we introduce a special class of formulas called polynomials — a linear analogue for conjunctive normal form.

A *monomial* is a product of atomic formulas and their negations.

A *polynomial* is a formula $a_1 + \cdots + a_n$, where $a_1, \ldots, a_n$ are monomials.

First we prove following special case of (iii)→(i).

Lemma 15 *For any polynomials a and b, if $[[a]]$ and $[[b]]$ coincide and $\sharp a = \sharp b$, then $a \overset{CLL^+}{\sim} b$.*

Example 5 $a{\bullet}b{+}c \overset{CLL^+}{\sim} a{+}b{\bullet}c$

Evidently $(a{\bullet}b{+}c)^\perp = c^\perp{\bullet}(b^\perp{+}a^\perp)$

and $(a{+}b{\bullet}c)^\perp = (c^\perp{+}b^\perp){\bullet}a^\perp$.

$$\dfrac{\dfrac{\Rightarrow b^\perp \ ; \ b \qquad \dfrac{\Rightarrow c \ ; \ c^\perp \qquad \Rightarrow b^\perp{+}a^\perp \ ; \ a{\bullet}b}{\Rightarrow c \ ; \ c^\perp{\bullet}(b^\perp{+}a^\perp) \ ; \ a{\bullet}b}(\bullet)}{\dfrac{\Rightarrow b^\perp \ ; \ b{\bullet}c \ ; \ c^\perp{\bullet}(b^\perp{+}a^\perp) \ ; \ a{\bullet}b}{\Rightarrow c^\perp{\bullet}(b^\perp{+}a^\perp) \ ; \ a{\bullet}b \ ; \ b^\perp \ ; \ b{\bullet}c}(\mathrm{ROT})}(\bullet)}{\Rightarrow c^\perp{\bullet}(b^\perp{+}a^\perp) \ ; \ a{\bullet}b{+}b^\perp{+}b{\bullet}c}(+)$$

$$\dfrac{\dfrac{\Rightarrow b{\bullet}c \ ; \ (c^\perp{+}b^\perp) \qquad \dfrac{\Rightarrow a^\perp \ ; \ a \qquad \Rightarrow b \ ; \ b^\perp}{\Rightarrow a^\perp \ ; \ a{\bullet}b \ ; \ b^\perp}(\bullet)}{\dfrac{\Rightarrow b{\bullet}c \ ; \ (c^\perp{+}b^\perp){\bullet}a^\perp \ ; \ a{\bullet}b \ ; \ b^\perp}{\Rightarrow (c^\perp{+}b^\perp){\bullet}a^\perp \ ; \ a{\bullet}b \ ; \ b^\perp \ ; \ b{\bullet}c}(\mathrm{ROT})}(\bullet)}{\Rightarrow (c^\perp{+}b^\perp){\bullet}a^\perp \ ; \ a{\bullet}b{+}b^\perp{+}b{\bullet}c}(+)$$

Proof of Lemma 15. Let $[[a]] \equiv [[b]]$. Then the polynomials a and b contain in the same order the same atomic formulas and their negations. They may differ only by occurrences of connectives $\bullet$ and $+$. Further, if also $\sharp a = \sharp b$, then the total number of connectives $\bullet$ in formula a is equal to that of formula b. Now the lemma is an easy consequence of Example 5 and the replacment property for $\sim$ in CLL^+. ■

Lemma 16 *For any formula c, there exists a polynomial c', such that*

$$[[c']] \equiv [[c]] \ \ , \ \ \sharp c' = \sharp c \ \ \text{and} \ \ CLL^+ \vdash c \Rightarrow c' \ \ .$$

Proof. Polynomial c' is obtained from c by skipping all parentheses. The proof is obvious from Example 6. ■

Example 6 We show that $CLL^+ \vdash a{\bullet}(b{+}c) \Rightarrow a{\bullet}b{+}c$.

Evidently $(a{\bullet}(b{+}c))^{\perp} = c^{\perp}{\bullet}b^{\perp}{+}a^{\perp}$.

$$\dfrac{\dfrac{\dfrac{\Rightarrow c\ ;\ c^{\perp} \qquad \dfrac{\Rightarrow a^{\perp}\ ;\ a{\bullet}b\ ;\ b^{\perp}}{\Rightarrow b^{\perp}\ ;\ a^{\perp}\ ;\ a{\bullet}b}(\mathrm{P})}{\Rightarrow c\ ;\ c^{\perp}{\bullet}b^{\perp}\ ;\ a^{\perp}\ ;\ a{\bullet}b}(\bullet)}{\Rightarrow c^{\perp}{\bullet}b^{\perp}\ ;\ a^{\perp}\ ;\ a{\bullet}b\ ;\ c}(\mathrm{P})}{\Rightarrow c^{\perp}{\bullet}b^{\perp}{+}a^{\perp}\ ;\ a{\bullet}b{+}c}(+)$$

Proof of Theorem 3 (iii)$\to$(i) . Let $\sharp a = \sharp b$ and $[[a]] = [[b]]$ in the free group. Using Lemma 16, we find corresponding polynomials a' and b', such that $CLL^+ \vdash a \Rightarrow a'$, $CLL^+ \vdash b \Rightarrow b'$, $\sharp a' = \sharp b'$ and $[[a']] = [[b']]$ in the free group. Obviously any join for a' and b' is also a join for a and b. So it remains to prove $a' \overset{CLL^+}{\sim} b'$.

Repeating the argument of Lemma 13, we reduce the proof to the cases

$$[[a']] \equiv \delta p p^{-1} \epsilon \ , \ \ [[b']] \equiv \delta \epsilon \ \ , \text{where} \ \ \ \delta \not\equiv \Lambda \ \ \text{or} \ \ \epsilon \not\equiv \Lambda$$

and

$$[[a']] \equiv \delta p^{-1} p \epsilon \ , \ \ [[b']] \equiv \delta \epsilon \ \ , \text{where} \ \ \ \delta \not\equiv \Lambda \ \ \text{or} \ \ \epsilon \not\equiv \Lambda \ \ .$$

We leave the dual proof of the latter to the reader.

Let for instance $[[a']] \equiv \delta p p^{-1} \epsilon$ and $\delta \not\equiv \Lambda$.

Now $\delta \equiv \gamma q$ or $\delta \equiv \gamma q^{-1}$, where $\gamma \in \Gamma^*$, $q \in \mathbf{Atom}$. Suppose $\delta \equiv \gamma q$. Let $b'(q\,|\,q{\bullet}p{+}p^{\perp})$ stand for the result of substituting $q{\bullet}p{+}p^{\perp}$ for the distinguished occurrence of atomic formula q in b'. It is easy to see that $CLL^+ \vdash q \Rightarrow q{\bullet}p{+}p^{\perp}$ and hence

$$CLL^+ \vdash b' \Rightarrow b'(q\,|\,q{\bullet}p{+}p^{\perp}) \ \ .$$

Evidently $[[b'(q\,|\,q{\bullet}p{+}p^{\perp})]] \equiv [[a']]$. Using Lemma 6 we get $\sharp b'(q\,|\,q{\bullet}p{+}p^{\perp}) = \sharp b'$.
By Lemma 15 we have $a' \overset{CLL^+}{\sim} b'(q\,|\,q{\bullet}p{+}p^{\perp})$ and consequently $a' \overset{CLL^+}{\sim} b'$.

In the similar way the rest follows from

$$CLL^+ \vdash q^\perp \Rightarrow q^\perp \bullet p + p^\perp \ ,$$

$$CLL^+ \vdash q \Rightarrow p + p^\perp \bullet q \ \text{ and}$$

$$CLL^+ \vdash q^\perp \Rightarrow p + p^\perp \bullet q^\perp \ .$$

This completes the proof of Theorem 3. ■

Theorem 4 *Let a and b be any two formulas in the language $\bullet$, $+$, $(\ )^\perp$. Then following three clauses are equivalent.*

(i) $a \overset{LL^+}{\sim} b$

(ii) $a \overset{LL^*}{\sim} b$

(iii) $\sharp a = \sharp b$ *and* $\forall p \in \mathbf{Atom}\ \#_p a = \#_p b$

Again (i)→(ii) is obvious, (ii)→(iii) follows from soundness results (cf. Lemma 6 (ii), Lemma 4 (ii)) and (iii)→(i) combines elements of proofs of Theorem 3 and Theorem 2.

7 Conservativity of linear logic over Lambek calculus

We introduce left and right implications in linear logic as abbreviations defined as

$$a \backslash b \rightleftharpoons a^\perp + b \ \text{ and } \ b/a \rightleftharpoons b + a^\perp \ .$$

It is easy to see that ordinary linear logic is conservative over non-directed Lambek calculus L^*P (by cut-elimination and noting that every sequent derivable in linear logic has exactly one formula in its succedent). Likewise it may be proved that L^+, L^+P and L^* lie faithfully embedded in CLL^+, LL^+ and CLL^* respectively.

In this section we give an explicit proof of conservativity of CLL^+ over L^+.

Let $\mathbf{Fm}(\backslash, /, \bullet)$ stand for the set of all linear formulas containing only $\bullet$, $\backslash$ and $/$. If one expresses these formulas using connectives $\bullet$, $+$ and $(\ )^\perp$, then the class $\mathbf{Fm}(\backslash, /, \bullet)$ can be recursively defined thus:

- $p \in \mathbf{Fm}(\backslash, /, \bullet)$ for any $p \in \mathbf{Atom}$,
- if $a \in \mathbf{Fm}(\backslash, /, \bullet)$ and $b \in \mathbf{Fm}(\backslash, /, \bullet)$ then $a \bullet b \in \mathbf{Fm}(\backslash, /, \bullet)$, $a^\perp + b \in \mathbf{Fm}(\backslash, /, \bullet)$ and $b + a^\perp \in \mathbf{Fm}(\backslash, /, \bullet)$.

First we formulate an easy lemma.

Lemma 17 *If $a \in \mathbf{Fm}(\backslash, /, \bullet)$ then $\sharp a = 1$.*

The proof is straightforward by induction on construction of formula a.

Theorem 5 *Let* $a_1, \ldots, a_n, b \in$ $\mathbf{Fm}(\backslash, /, \bullet)$ *. Then* $L^+ \vdash a_1, \ldots, a_n \Rightarrow b$ *if and only if* $CLL^+ \vdash a_1 \bullet \cdots \bullet a_n \Rightarrow b$

Proof. Let $(a_1, \ldots, a_n)^\perp$ stand for ${a_n}^\perp; \ldots; {a_1}^\perp$. Note that

$$CLL^+ \vdash a_1 \bullet \cdots \bullet a_n \Rightarrow b \ \text{ means } \ CLL^+ \vdash \ \Rightarrow {a_n}^\perp + \cdots + {a_1}^\perp \ ; \ b$$

and latter is equivalent to $CLL^+ \vdash \ \Rightarrow {a_n}^\perp \ ; \ \ldots \ ; \ {a_1}^\perp \ ; \ b$.

Hence, it suffices to prove that

$$L^+ \vdash X \Rightarrow d \ \text{ iff } \ CLL^+ \vdash \ \Rightarrow X^\perp \ ; \ d \ \ .$$

$\boxed{\text{Proof of } \longrightarrow}$ Induction on derivations in L^+.

Case $a \Rightarrow a$: The sequent $\Rightarrow a^\perp \ ; \ a$ is an axiom of CLL^+.

Case $(\Rightarrow \backslash)$: Let the last rule be

$$\frac{a, X \Rightarrow b}{X \Rightarrow a \backslash b}(\Rightarrow \backslash)$$

By IH we have $CLL^+ \vdash \ \Rightarrow (a, X)^\perp \ ; \ b$. We note that $(a, X)^\perp$ is the same as $X^\perp \ ; \ a^\perp$. Now, we derive in CLL^+

$$\frac{\Rightarrow X^\perp \ ; \ a^\perp \ ; \ b}{\Rightarrow X^\perp \ ; \ a^\perp + b}(+)$$

Case $(\backslash \Rightarrow)$:

$$\frac{X \Rightarrow a \qquad Y \ , \ b \ , \ Z \Rightarrow c}{Y \ , \ X \ , \ a \backslash b \ , \ Z \Rightarrow c}(\backslash \Rightarrow) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Rightarrow Z^\perp \ ; \ b^\perp \ ; \ Y^\perp \ ; \ c}{\Rightarrow Y^\perp \ ; \ c \ ; \ Z^\perp \ ; \ b^\perp}(\mathrm{ROT}) \qquad \dfrac{\Rightarrow X^\perp \ ; \ a}{\Rightarrow a \ ; \ X^\perp}(\mathrm{ROT})}{\Rightarrow Y^\perp \ ; \ c \ ; \ Z^\perp \ ; \ b^\perp \bullet a \ ; \ X^\perp}(\bullet)}{\Rightarrow Z^\perp \ ; \ b^\perp \bullet a \ ; \ X^\perp \ ; \ Y^\perp \ ; \ c}(\mathrm{ROT})$$

Case $(\Rightarrow \bullet)$:

$$\frac{X \Rightarrow a \qquad Y \Rightarrow b}{X, Y \Rightarrow a \bullet b}(\Rightarrow \bullet) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Rightarrow X^\perp \ ; \ a \qquad \dfrac{\Rightarrow Y^\perp \ ; \ b}{\Rightarrow b \ ; \ Y^\perp}(\mathrm{ROT})}{\Rightarrow X^\perp \ ; \ a \bullet b \ ; \ Y^\perp}(\bullet)}{\Rightarrow Y^\perp \ ; \ X^\perp \ ; \ a \bullet b}(\mathrm{ROT})$$

Case $(\bullet \Rightarrow)$:

$$\frac{X \ , \ a \ , \ b \ , \ Y \Rightarrow c}{X \ , \ a \bullet b \ , \ Y \Rightarrow c}(\bullet \Rightarrow) \quad \rightsquigarrow \quad \frac{\Rightarrow Y^\perp \ ; \ b^\perp \ ; \ a^\perp \ ; \ X^\perp \ ; \ c}{\Rightarrow Y^\perp \ ; \ b^\perp + a^\perp \ ; \ X^\perp \ ; \ c}(+)$$

We omit the dual cases of $(\Rightarrow /)$ and $(/ \Rightarrow)$.

$\boxed{\text{Proof of } \longleftarrow}$ We prove by induction on derivations in CLL^+ that

$$\text{if } \ d \in \mathbf{Fm}(\backslash, /, \bullet) \quad , \ Z, W \subset \mathbf{Fm}(\backslash, /, \bullet) \quad \text{and} \ CLL^+ \vdash \ \Rightarrow Z^\perp \ ; \ d \ ; \ W^\perp$$

$$\text{then} \ \ L^+ \vdash Z, W \Rightarrow d \ \ .$$

The obvious cases of axioms and rotation are omitted.

Case 1a: The last rule is $(+)$, the main formula is d.

We shall consider the possibility $d = a^\perp + b$ and omit the dual case $d = b + a^\perp$.

$$\frac{\Rightarrow Z^\perp \ ; \ a^\perp \ ; \ b \ ; \ W^\perp}{\Rightarrow Z^\perp \ ; \ a^\perp + b \ ; \ W^\perp}(+) \qquad \rightsquigarrow \qquad \frac{a, Z, W \Rightarrow b}{Z, W \Rightarrow a \backslash b}(\Rightarrow \backslash)$$

Case 1b: The last rule is $(+)$, d is not the main formula.

$$\frac{\Rightarrow Z^\perp \ ; \ {W_1}^\perp \ ; \ a^\perp \ ; \ b^\perp \ ; \ {W_2}^\perp}{\Rightarrow Z^\perp \ ; \ {W_1}^\perp \ ; \ a^\perp + b^\perp \ ; \ {W_2}^\perp}(+) \qquad \rightsquigarrow \qquad \frac{Z \ , \ b \ , \ a \ , \ Y \Rightarrow d}{Z \ , \ b \bullet a \ , \ Y \Rightarrow d}(\Rightarrow \backslash)$$

Case 2a: The last rule is $(\bullet)$, the main formula is d.

$$\frac{\Rightarrow Z^\perp \ ; \ a \qquad \Rightarrow b \ ; \ W^\perp}{\Rightarrow Z^\perp \ ; \ a \bullet b \ ; \ W^\perp}(\bullet) \qquad \rightsquigarrow \qquad \frac{Z \Rightarrow a \qquad W \Rightarrow b}{Z, W \Rightarrow a \bullet b}(\Rightarrow \bullet)$$

Case 2b: The last rule is $(\bullet)$, d is not the main formula.

Now the main formula lies in $Z^\perp$ or in $W^\perp$. So its negation belongs to $\mathbf{Fm}(\backslash, /, \bullet)$ and thus the main formula itself must be of the form $a \bullet b^\perp$ or $b^\perp \bullet a$. The choices between $Z^\perp$ and $W^\perp$, and between $a \bullet b^\perp$ and $b^\perp \bullet a$ on the other hand, produce four subcases, two of which we omit for the reason of duality.

Case 2b.i: $a \bullet b^\perp \in Z^\perp$

$$\frac{\Rightarrow {Z_1}^\perp \ ; \ a \qquad \Rightarrow b^\perp \ ; \ {Z_2}^\perp \ ; \ d \ ; \ W^\perp}{\Rightarrow {Z_1}^\perp \ ; \ a \bullet b^\perp \ ; \ {Z_2}^\perp \ ; \ d \ ; \ W^\perp}(\bullet) \qquad \rightsquigarrow \qquad \frac{Z_1 \Rightarrow a \qquad Z_2 \ , \ b \ , \ W \Rightarrow d}{Z_2 \ , \ b/a \ , \ Z_1 \ , \ W \Rightarrow d}(/ \Rightarrow)$$

Case 2b.ii: $b^\perp \bullet a \in Z^\perp$

This is the crucial point of the proof. We show that this case cannot occur in CLL^+-derivations.

$$\frac{\Rightarrow {Z_1}^\perp \ ; \ b^\perp \qquad \Rightarrow a \ ; \ {Z_2}^\perp \ ; \ d \ ; \ W^\perp}{\Rightarrow {Z_1}^\perp \ ; \ b^\perp \bullet a \ ; \ {Z_2}^\perp \ ; \ d \ ; \ W^\perp}(\bullet)$$

By Lemma 17 we have

$$\sharp b = 1 \ \text{ and } \ \forall z \in Z_1 \ \sharp z = 1 \quad .$$

Using Lemma 5, we obtain

$$\sharp(\Rightarrow {Z_1}^\perp \ ; \ b^\perp) = 0 \quad ,$$

and Lemma 6 yields a contradiction with $CLL^+ \vdash \ \Rightarrow {Z_1}^\perp \ ; \ b^\perp$.

We have proved that L^+ lies faithfully embedded in CLL^+. ■

Three other clauses have similar proofs.

8 Product-free fragments of Lambek calculus

Let $\mathbf{Fm}(\backslash,\ /)$ stand for the set of all types in the language of two implications. In this section we show that a pair of types has a join in $\mathbf{Fm}(\backslash,\ /,\ \bullet)$ if and only if it has a join from $\mathbf{Fm}(\backslash,\ /)$. We also note that the diamond property for product-free fragment is different from the one in full Lambek calculus.

8.1 Product-free joins

Theorem 6 (i) *Theorem 1 holds for* $L^{+}(\backslash, /)$ *and* $L^{*}(\backslash, /)$*.*

(ii) *Theorem 2 holds for* $L^{+}P(\rightarrow)$ *and* $L^{*}P(\rightarrow)$*.*

Here $L^{+}(\backslash, /)$ denotes the fragment of L^{+} without product. Similarly for $L^{*}(\backslash, /)$, $L^{+}P(\rightarrow)$ and $L^{*}P(\rightarrow)$.

PROOF. All the cases are similar. We give a proof for $L^{+}(\backslash, /)$ only. It suffices to prove that, if $a, b \in \mathbf{Fm}(\backslash,\ /)$ and $[[a]] = [[b]]$ in the free group, then in $\mathbf{Fm}(\backslash,\ /)$ there is a join for a and b. Using Theorem 1 we get a join $c \in \mathbf{Fm}(\backslash,\ /,\ \bullet)$. Lemma 18 gives a type $c' \in \mathbf{Fm}(\backslash,\ /)$ such that $L^{+} \vdash a \Rightarrow c'$ and $L^{+} \vdash b \Rightarrow c'$. Cut-elimination in Lambek calculus involves that L^{+} is conservative over $L^{+}(\backslash, /)$. Hence, $a \overset{L^{+}(\backslash, /)}{\sim} b$. ■

Lemma 18 *Let* $c \in \mathbf{Fm}(\backslash,\ /,\ \bullet)$ *. Then:*

(i) $\exists c' \in \mathbf{Fm}(\backslash,\ /)$ *such that* $L^{+} \vdash c \Rightarrow c'$,

(ii) $\exists X \subset \mathbf{Fm}(\backslash,\ /)$ *such that* $L^{+} \vdash X \Rightarrow c$.

PROOF. Induction on construction of type c.
CASE $c = p$: We take $c' = p$ and $X = p$.
CASE $c = a \bullet b$: Let $a \Rightarrow a'$, $b \Rightarrow b'$, $Y \Rightarrow a$, $Z \Rightarrow b$,
where $a', b' \in \mathbf{Fm}(\backslash,\ /)$ and $Y, Z \subset \mathbf{Fm}(\backslash,\ /)$.
We put $Z = X, Y$ and $c' = p/(b'\backslash(a'\backslash p))$ for some $p \in \mathbf{Atom}$.

$$\dfrac{\dfrac{\dfrac{b \Rightarrow b' \qquad \dfrac{a \Rightarrow a' \qquad p \Rightarrow p}{a\ ,\ a'\backslash p \Rightarrow p}\,(\backslash \Rightarrow)}{a\ ,\ b\ ,\ b'\backslash(a'\backslash p) \Rightarrow p}\,(\backslash \Rightarrow)}{a\ ,\ b \Rightarrow p/(b'\backslash(a'\backslash p))}\,(\Rightarrow /)}{a \bullet b \Rightarrow p/(b'\backslash(a'\backslash p))}\,(\bullet \Rightarrow) \qquad\qquad \dfrac{Y \Rightarrow a \qquad Z \Rightarrow b}{Y, Z \Rightarrow a \bullet b}\,(\Rightarrow \bullet)$$

CASE $c = a\backslash b$: Let $a \Rightarrow a'$, $b \Rightarrow b'$, $\hat{a}_1, \ldots, \hat{a}_n \Rightarrow a$, $\hat{b}_1, \ldots, \hat{b}_m \Rightarrow b$
for some $a', b', \hat{a}_i, \hat{b}_j \in \mathbf{Fm}(\backslash,\ /)$.

We put $c' = \hat{a}_n\backslash(\ldots\backslash(\hat{a}_1\backslash b')\ldots)$ and $X = a'\backslash\hat{b}_1\ ,\ \hat{b}_2,\ldots,\hat{b}_m$.

$$\cfrac{\cfrac{\cfrac{\hat{a}_1,\ldots,\hat{a}_n \Rightarrow a \qquad b \Rightarrow b'}{\hat{a}_1,\ldots,\hat{a}_n\ ,\ a\backslash b \Rightarrow b'}(\backslash\Rightarrow)}{\vdots}(\Rightarrow\backslash)}{a\backslash b \Rightarrow \hat{a}_n\backslash(\ldots\backslash(\hat{a}_1\backslash b')\ldots)}(\Rightarrow\backslash)
\qquad
\cfrac{\cfrac{a \Rightarrow a' \qquad \hat{b}_1,\ldots,\hat{b}_m \Rightarrow b}{a\ ,\ a'\backslash\hat{b}_1\ ,\ \hat{b}_2,\ldots,\hat{b}_m \Rightarrow b}(\backslash\Rightarrow)}{a'\backslash\hat{b}_1\ ,\ \hat{b}_2,\ldots,\hat{b}_m \Rightarrow a\backslash b}(\Rightarrow\backslash)$$

CASE $c = b/a$: Similar to the previous case. ■

8.2 Diamond property

Surprisingly, product-free fragment of Lambek calculus does not possess the diamond property (cf. Lemma 7). Instead of this a slightly different lemma holds.

Lemma 19 *Let* $a_1,\ldots,a_n \in$ **Fm**$(\backslash, /)$ *. Then following two assertions are equivalent.*

(i) $\exists b \in$ **Fm**$(\backslash, /)$ *such that* $\forall i \le n\ \ L^+(\backslash,/) \vdash a_i \Rightarrow b$,

(ii) *there is a sequence* X *consisting of* n *product-free types such that*

$$\forall i \le n\ \ L^+(\backslash,/) \vdash X \Rightarrow a_i\ .$$

Remark. For any n, there exist equivalent types $a_1,\ldots,a_n \in$ **Fm**$(\backslash, /)$ such that the sequence X in Lemma 19 cannot contain less than n types. Take for example $n = 2$, $a_1 = p/(p\backslash(p\backslash p))$ and $a_2 = q/(p\backslash(p\backslash q))$.
Obviously $L^+(\backslash,/) \vdash p,p \Rightarrow a_1$ and $L^+(\backslash,/) \vdash p,p \Rightarrow a_2$. Nevertheless, there is no product-free type d such that $L^+(\backslash,/) \vdash d \Rightarrow a_1$ and $L^+(\backslash,/) \vdash d \Rightarrow a_2$.

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