

# Noncommutative linear logic

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## 1 Formal languages

A *formal language* is a set of finite words over a finite alphabet.

**Example.** Consider the alphabet  $\Sigma = \{a, e, v\}$ . The set  $\{ve, veave, veaveave, veaveaveave, \dots\}$  is a formal language.

Two important approaches to formal language specification:

- Noam Chomsky (recursion-theoretic approach)
- Jim Lambek (logico-algebraic approach)  
J. Lambek, *The mathematics of sentence structure*,  
American Mathematical Monthly **65** (1958), no. 3, 154–170.

By  $\circ$  we denote the concatenation operator.  
 $\Sigma^*$  is the set of all words over the alphabet  $\Sigma$ .  
 $\Sigma^+$  is the set of all non-empty words over the alphabet  $\Sigma$ .

## 2 Lambek calculus

J. Lambek considers three basic operations on languages:

$$\begin{aligned}\mathcal{A} \cdot \mathcal{B} &\equiv \{x \circ y \mid x \in \mathcal{A}, y \in \mathcal{B}\}, \\ \mathcal{A} \setminus \mathcal{B} &\equiv \{y \in \Sigma^+ \mid \mathcal{A} \cdot \{y\} \subseteq \mathcal{B}\}, \\ \mathcal{B} / \mathcal{A} &\equiv \{x \in \Sigma^+ \mid \{x\} \cdot \mathcal{A} \subseteq \mathcal{B}\}.\end{aligned}$$

**Example.** Let  $\mathcal{A} = \{j, m\}$  and  $\mathcal{B} = \{je, jrj, jrm, me, mrj, mrm\}$ .  
Then  $\mathcal{A} \setminus \mathcal{B} = \{e, rj, rm\}$ .

**Definition.** *Types* are the elements of the minimal set  $\text{Tp}$  such that

- $\{p_0, p_1, p_2, \dots\} \subset \text{Tp}$
- If  $A \in \text{Tp}$  and  $B \in \text{Tp}$ , then  $(A \cdot B) \in \text{Tp}$ ,  $(A \setminus B) \in \text{Tp}$ , and  $(A/B) \in \text{Tp}$ .

Derivable objects of  $L_H$  are  $A \rightarrow B$ , where  $A \in \text{Tp}$  and  $B \in \text{Tp}$ .

**Axioms and rules of  $L_H$**

$$A \rightarrow A \quad (A \cdot B) \cdot C \rightarrow A \cdot (B \cdot C) \quad A \cdot (B \cdot C) \rightarrow (A \cdot B) \cdot C$$

$$\begin{array}{ccc} \frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C} & \frac{A \cdot B \rightarrow C}{A \rightarrow C/B} & \frac{A \cdot B \rightarrow C}{B \rightarrow A \setminus C} \\ & \frac{A \rightarrow C/B}{A \cdot B \rightarrow C} & \frac{B \rightarrow A \setminus C}{A \cdot B \rightarrow C} \end{array}$$

We write  $L_H \vdash A \rightarrow B$  for “ $A \rightarrow B$  is derivable in the calculus  $L_H$ ”.

**Example.** Let  $A, B \in \text{Tp}$ . Then  $L_H \vdash A \cdot (A \setminus B) \rightarrow B$ .

$$\frac{A \setminus B \rightarrow A \setminus B}{A \cdot (A \setminus B) \rightarrow B}$$

**Remark.** There exist  $A, B \in \text{Tp}$  such that  $L_H \not\vdash B \rightarrow A \cdot (A \setminus B)$ .

**Example.**  $A \cdot (B/C) \rightarrow (A \cdot B)/C$  is derivable in  $L_H$ .

$$\frac{\frac{\frac{B/C \rightarrow B/C}{(B/C) \cdot C \rightarrow B} \quad \frac{A \cdot B \rightarrow A \cdot B}{B \rightarrow A \setminus (A \cdot B)}}{(B/C) \cdot C \rightarrow A \setminus (A \cdot B)} \quad \frac{(A \cdot (B/C)) \cdot C \rightarrow A \cdot ((B/C) \cdot C) \quad A \cdot ((B/C) \cdot C) \rightarrow A \cdot B}{(A \cdot (B/C)) \cdot C \rightarrow A \cdot B}}{A \cdot (B/C) \rightarrow (A \cdot B)/C}$$

**Definition.**  $A \stackrel{L_H}{\leftrightarrow} B$  iff  $L_H \vdash A \rightarrow B$  and  $L_H \vdash B \rightarrow A$ .

**Example.**

$$\begin{aligned} (A \setminus B)/C &\stackrel{L_H}{\leftrightarrow} A \setminus (B/C), \\ A/(B \cdot C) &\stackrel{L_H}{\leftrightarrow} (A/C)/B, \\ A \cdot (A \setminus (A \cdot B)) &\stackrel{L_H}{\leftrightarrow} A \cdot B. \end{aligned}$$

**Example.**

$$\begin{aligned} L_H \vdash ((B/A) \setminus C) \setminus D &\rightarrow (B \setminus C) \setminus (A \setminus D), \\ L_H \not\vdash ((A \setminus B) \setminus C) \setminus D &\rightarrow C \setminus ((B \setminus A) \setminus D). \end{aligned}$$

### 3 Lambek calculus L with sequents

Derivable objects of the calculus L are *sequents*  $\Gamma \rightarrow A$ , where  $A \in \text{Tp}$  and  $\Gamma \in \text{Tp}^+$ .

**Axioms and rules of L**

$$\begin{array}{l} A \rightarrow A \\ \frac{A \Pi \rightarrow B}{\Pi \rightarrow A \setminus B} \ (\rightarrow \setminus), \text{ where } \Pi \neq \Lambda \\ \frac{\Pi A \rightarrow B}{\Pi \rightarrow B/A} \ (\rightarrow /), \text{ where } \Pi \neq \Lambda \\ \frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma \Delta \rightarrow A \cdot B} \ (\rightarrow \cdot) \end{array} \quad \begin{array}{l} \frac{\Phi \rightarrow B \quad \Gamma B \Delta \rightarrow A}{\Gamma \Phi \Delta \rightarrow A} \ (\text{cut}) \\ \frac{\Phi \rightarrow A \quad \Gamma B \Delta \rightarrow C}{\Gamma \Phi (A \setminus B) \Delta \rightarrow C} \ (\setminus \rightarrow) \\ \frac{\Phi \rightarrow A \quad \Gamma B \Delta \rightarrow C}{\Gamma (B/A) \Phi \Delta \rightarrow C} \ (/ \rightarrow) \\ \frac{\Gamma A B \Delta \rightarrow C}{\Gamma (A \cdot B) \Delta \rightarrow C} \ (\cdot \rightarrow) \end{array}$$

Here  $\Lambda$  is the empty sequence,  $A, B, C \in \text{Tp}$ , and  $\Gamma, \Delta, \Phi, \Pi \in \text{Tp}^*$ .

**Theorem 1 (J. Lambek, 1958).**  $L \vdash A_1 \dots A_n \rightarrow B$  if and only if  $L_H \vdash A_1 \cdot \dots \cdot A_n \rightarrow B$ .

**Cut-elimination theorem (J. Lambek, 1958).** A sequent is derivable in L if and only if it is derivable in L without (cut).

**Example.**  $L \vdash A \cdot (B/C) \rightarrow (A \cdot B)/C$

$$\begin{array}{l} \frac{C \rightarrow C \quad B \rightarrow B}{(B/C) C \rightarrow B} \ (/ \rightarrow) \\ \frac{A \rightarrow A \quad (B/C) C \rightarrow B}{A (B/C) C \rightarrow (A \cdot B)} \ (\rightarrow \cdot) \\ \frac{A (B/C) C \rightarrow (A \cdot B)}{A (B/C) \rightarrow (A \cdot B)/C} \ (\rightarrow /) \\ \frac{A (B/C) \rightarrow (A \cdot B)/C}{A \cdot (B/C) \rightarrow (A \cdot B)/C} \ (\cdot \rightarrow) \end{array}$$

**Remark.**  $L \not\vdash (A \cdot B)/C \rightarrow A \cdot (B/C)$ .

## 4 Grammars

**Definition.** A *Lambek categorial grammar* is a triple  $\langle \Sigma, D, f \rangle$  such that  $|\Sigma| < \infty$ ,  $D \in \text{Tp}$ ,  $f: \Sigma \rightarrow \mathcal{P}(\text{Tp})$ , and  $|f(t)| < \infty$  for each  $t \in \Sigma$ .

The grammar *recognizes* the language

$$\mathcal{L}_L(\Sigma, D, f) \rightleftharpoons \{t_1 \dots t_n \in \Sigma^+ \mid \exists B_1 \in f(t_1) \dots \exists B_n \in f(t_n) \\ \text{L} \vdash B_1 \dots B_n \rightarrow D\}$$

**Example.**

$$\begin{aligned} np = p_1 \quad s = p_2 \quad D = s \quad \Sigma = \{\text{John, Mary, works, recommends}\} \\ f(\text{John}) = f(\text{Mary}) = \{np\} \\ f(\text{works}) = \{(np \backslash s)\} \\ f(\text{recommends}) = \{((np \backslash s)/np)\} \\ \frac{np \rightarrow np \quad \frac{np \rightarrow np \quad s \rightarrow s}{np (np \backslash s) \rightarrow s} (\backslash \rightarrow)}{np \quad ((np \backslash s)/np) \quad np \rightarrow s} (/ \rightarrow) \\ \text{John} \quad \text{recommends} \quad \text{Mary} \end{aligned}$$

B. Carpenter, *Type-Logical Semantics*, MIT Press, Cambridge, MA, 1997.  
<http://www.colloquial.com/tlg/parser.html>

**Example.**

$$\begin{aligned} \Sigma = \{\text{Val, recommends, he, she, him, her}\} \\ f(\text{Val}) = \{np\} \\ f(\text{recommends}) = \{((np \backslash s)/np)\} \\ f(\text{he}) = f(\text{she}) = \{(s/(np \backslash s))\} \\ f(\text{him}) = f(\text{her}) = \{((s/np) \backslash s)\} \\ \frac{np \rightarrow np \quad \frac{(np \backslash s) \rightarrow (np \backslash s) \quad s \rightarrow s}{(s/(np \backslash s)) (np \backslash s) \rightarrow s} (/ \rightarrow)}{(s/(np \backslash s)) ((np \backslash s)/np) np \rightarrow s} (/ \rightarrow) \\ \frac{(s/(np \backslash s)) ((np \backslash s)/np) \rightarrow (s/np)}{(s/(np \backslash s)) ((np \backslash s)/np) ((s/np) \backslash s) \rightarrow s} (\rightarrow /)}{(s/(np \backslash s)) ((np \backslash s)/np) ((s/np) \backslash s) \rightarrow s} (\backslash \rightarrow) \\ \text{She} \quad \text{recommends} \quad \text{him} \end{aligned}$$

**Example.**

$\Sigma = \{\text{John, Val, succeeds, exists, helps, recommends, student, professor, club, a, the, every, this, strange, whenever, whom, relatively, everywhere, or}\}$

John succeeds whenever Val recommends a club or helps the student whom this relatively strange professor recommends.

$$\begin{aligned} f(\text{Val}) = \{np\} \\ f(\text{succeeds}) = f(\text{exists}) = \{(np \backslash s)\} \\ f(\text{helps}) = f(\text{recommends}) = \{((np \backslash s)/np)\} \\ f(\text{student}) = f(\text{professor}) = f(\text{club}) = \{n\} \\ f(\text{a}) = f(\text{the}) = f(\text{every}) = \{(np/n)\} \\ f(\text{this}) = \{(np/n), np\} \\ f(\text{strange}) = \{(n/n)\} \\ f(\text{whenever}) = \{((s \backslash s)/s)\} \\ f(\text{whom}) = \{((n \backslash n)/(s/np))\} \\ f(\text{relatively}) = \{((n/n)/(n/n))\} \\ f(\text{everywhere}) = \{((np \backslash s) \backslash (np \backslash s))\} \\ f(\text{or}) = \{((np \backslash np)/np), ((s \backslash s)/s), ((np \backslash s) \backslash (np \backslash s))/(np \backslash s)\} \end{aligned}$$

**Definition.** A *context-free grammar* is a 4-tuple  $\langle \Sigma, \mathcal{W}, S, \mathcal{R} \rangle$  such that  $|\Sigma| < \infty$ ,  $|\mathcal{W}| < \infty$ ,  $\Sigma \cap \mathcal{W} = \emptyset$ ,  $S \in \mathcal{W}$ ,  $\mathcal{R} \subset \{A \mapsto u \mid A \in \mathcal{W} \text{ and } u \in (\Sigma \cup \mathcal{W})^+\}$ , and  $|\mathcal{R}| < \infty$ .  
The grammar *recognizes* the language

$$\mathcal{G}(\Sigma, \mathcal{W}, S, \mathcal{R}) \doteq \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R}) \cap \Sigma^+.$$

Here  $\bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$  is defined inductively.

- $S \in \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$
- If  $u_1, u_2, u_3 \in (\Sigma \cup \mathcal{W})^*$ ,  $A \in \mathcal{W}$ ,  $u_1 A u_3 \in \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$ , and  $A \mapsto u_2 \in \mathcal{R}$ , then  $u_1 u_2 u_3 \in \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$ .

**Example.**

$$\Sigma = \{\text{John, Mary, works, recommends}\} \quad \mathcal{W} = \{S, NP, VP, V_t\}$$

$$\mathcal{R} = \{S \mapsto NP VP, \quad VP \mapsto V_t NP, \quad NP \mapsto \text{John}, \\ NP \mapsto \text{Mary}, \quad VP \mapsto \text{works}, \quad V_t \mapsto \text{recommends}\}$$

**Theorem 2 (J. M. Cohen, 1967).**

$$\forall \langle \Sigma, \mathcal{W}, S, \mathcal{R} \rangle \exists D \exists f \text{ such that } \mathcal{L}_L(\Sigma, D, f) = \mathcal{G}(\Sigma, \mathcal{W}, S, \mathcal{R})$$

**Theorem 3 (1992).**

$$\forall \langle \Sigma, D, f \rangle \exists \mathcal{W} \exists S \exists \mathcal{R} \text{ such that } \mathcal{G}(\Sigma, \mathcal{W}, S, \mathcal{R}) = \mathcal{L}_L(\Sigma, D, f)$$

**Definition.**

$$\|p_i\| \doteq 1, \\ \|A \cdot B\| = \|A \setminus B\| = \|A/B\| \doteq \|A\| + \|B\|.$$

Proof of Theorem 3.

$$m \doteq \max(\|D\|, \max_{t \in \Sigma} \max_{B \in f(t)} \|B\|)$$

$$\mathcal{W} \doteq \{A \in \text{Tp} \mid \|A\| \leq m\} \\ S \doteq D \\ \mathcal{R} \doteq \{B \mapsto t \mid t \in \Sigma \text{ and } B \in f(t)\} \cup \\ \cup \{C \mapsto AB \mid A, B, C \in \mathcal{W} \text{ and } L \vdash AB \rightarrow C\} \cup \\ \cup \{D \mapsto A \mid A \in \mathcal{W} \text{ and } L \vdash A \rightarrow D\}$$

□

**Example.**

$$\Sigma = \{\text{John, Mary, recommends}\}$$

$$np \mapsto \text{John} \in \mathcal{R} \\ np \mapsto \text{Mary} \in \mathcal{R} \\ ((np \setminus s)/np) \mapsto \text{recommends} \in \mathcal{R} \\ s \mapsto np \quad (np \setminus s) \in \mathcal{R} \\ (np \setminus s) \mapsto ((np \setminus s)/np) \quad np \in \mathcal{R} \\ \text{etc.}$$

Theorem 3 follows from Lemma 1.

**Lemma 1.** If  $L \vdash B_1 \dots B_n \rightarrow D$ , where  $n \geq 2$ ,  $\|D\| \leq m$ , and  $\|B_i\| \leq m$  for each  $i$ , then  $B_1 \dots B_n \rightarrow D$  follows by means of the cut rule from  $n-1$  derivable sequents of the form  $A_1 A_2 \rightarrow A_3$ , where  $\|A_j\| \leq m$  for each  $j$ .

## 5 Language models

**Definition.** A *language model* (*free semigroup model*) is a pair  $\langle \Sigma^+, v \rangle$  such that  $\Sigma$  is a finite or countable alphabet and

- $v(p_i) \subseteq \Sigma^+$ ,
- $v(A \cdot B) = v(A) \cdot v(B)$ ,
- $v(A \setminus B) = v(A) \setminus v(B) = \{y \in \Sigma^+ \mid v(A) \cdot \{y\} \subseteq v(B)\}$ ,
- $v(B/A) = v(B)/v(A) = \{x \in \Sigma^+ \mid \{x\} \cdot v(A) \subseteq v(B)\}$ .

**Remark.** L is sound with respect to language models.

**Definition.**  $L(\setminus, /)$  is the elementary fragment of L without  $\cdot$ .

**Remark.** L is conservative over  $L(\setminus, /)$ .

**Remark (W. Buszkowski, 1982).**  $L(\setminus, /)$  is complete with respect to language models.

Proof.

$$\begin{aligned} \Sigma &\equiv \text{Tp} \\ v(A) &\equiv \{\Gamma \in \text{Tp}^+ \mid L \vdash \Gamma \rightarrow A\} \end{aligned}$$

□

**Theorem 4 (1993).** A *sequent* is derivable in L if and only if it is true in every language model.

**Example.** Let  $p, q \in \text{Pr}$ . Then  $L \not\vdash p \rightarrow p \cdot (q \setminus q)$ .

$$\begin{aligned} \Sigma &= \{a_1, a_2\} & v(p) &= \{a_1\} \\ & & v(q) &= \{a_2\} \end{aligned}$$

$$\begin{aligned} v(q \setminus q) &= \emptyset \\ v(p \cdot (q \setminus q)) &= \emptyset \\ v(p) &= \{a_1\} \not\subseteq \emptyset = v(p \cdot (q \setminus q)) \end{aligned}$$

**Example.** Let  $p, q, r \in \text{Pr}$ . Then  $L \not\vdash (p \cdot q)/r \rightarrow p \cdot (q/r)$ .

$$\begin{aligned} \Sigma &= \{a_1, a_2, a_3\} & v(p) &= \{a_1 a_2\} \\ & & v(q) &= \{a_3\} \\ & & v(r) &= \{a_2 a_3\} \\ \\ v(p \cdot q) &= \{a_1 a_2 a_3\} \\ v((p \cdot q)/r) &= \{a_1\} \\ v(q/r) &= \emptyset \\ v(p \cdot (q/r)) &= \emptyset \\ v((p \cdot q)/r) &= \{a_1\} \not\subseteq \emptyset = v(p \cdot (q/r)) \end{aligned}$$

**Example.**

$$\begin{aligned} \Sigma' &= \{b, c\} & v'(p) &= \{bcbbccb\} \\ & & v'(q) &= \{bcccb\} \\ & & v'(r) &= \{bccbbcccb\} \end{aligned}$$

**Corollary 1.** A *sequent* is derivable in L if and only if it is true in every language model over a two-symbol alphabet.

Proof. Let  $\Sigma = \{a_1, a_2, \dots\}$ . Put  $\Sigma' = \{b, c\}$ .

Map  $a_i$  to  $\underbrace{bcc \dots cb}_i$ .

□

## 6 The calculus $L^*$

Derivable objects of the calculus  $L^*$  are *sequents*  $\Gamma \rightarrow A$ , where  $A \in \text{Tp}$  and  $\Gamma \in \text{Tp}^*$ .

**Axioms and rules of  $L^*$**

$$\begin{array}{l}
 A \rightarrow A \\
 \frac{A \Pi \rightarrow B}{\Pi \rightarrow A \setminus B} (\rightarrow \setminus) \\
 \frac{\Pi A \rightarrow B}{\Pi \rightarrow B/A} (\rightarrow /) \\
 \frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma \Delta \rightarrow A \cdot B} (\rightarrow \cdot)
 \end{array}
 \qquad
 \begin{array}{l}
 \frac{\Phi \rightarrow B \quad \Gamma B \Delta \rightarrow A}{\Gamma \Phi \Delta \rightarrow A} (\text{cut}) \\
 \frac{\Phi \rightarrow A \quad \Gamma B \Delta \rightarrow C}{\Gamma \Phi (A \setminus B) \Delta \rightarrow C} (\setminus \rightarrow) \\
 \frac{\Phi \rightarrow A \quad \Gamma B \Delta \rightarrow C}{\Gamma (B/A) \Phi \Delta \rightarrow C} (/ \rightarrow) \\
 \frac{\Gamma A B \Delta \rightarrow C}{\Gamma (A \cdot B) \Delta \rightarrow C} (\cdot \rightarrow)
 \end{array}$$

**Example.**

$$\frac{A \rightarrow A \quad \frac{B \rightarrow B}{\rightarrow B \setminus B} (\rightarrow \setminus)}{A \rightarrow A \cdot (B \setminus B)} (\rightarrow \cdot)$$

**Remark.**  $L^* \vdash A \rightarrow A \cdot (B \setminus B)$ , but  $L \not\vdash A \rightarrow A \cdot (B \setminus B)$ .

**Cut-elimination theorem.** We may drop (cut).

**Definition.** A *free monoid model* is a pair  $\langle \Sigma^*, v \rangle$  such that  $\Sigma$  is a finite or countable alphabet and

- $v(p_i) \subseteq \Sigma^*$ ,
- $v(A \cdot B) = v(A) \cdot v(B)$ ,
- $v(A \setminus B) = \{y \in \Sigma^* \mid v(A) \cdot \{y\} \subseteq v(B)\}$ ,
- $v(B/A) = \{x \in \Sigma^* \mid \{x\} \cdot v(A) \subseteq v(B)\}$ .

**Theorem 5 (1996).** *A sequent is derivable in  $L^*$  if and only if it is true in every free monoid model.*

## 7 Cyclic linear logic MCLL

We consider only multiplicative fragments of linear logic calculi.

D. N. Yetter, *Quantales and noncommutative linear logic*, Journal of Symbolic Logic, **55** (1990), no. 1, pp. 41–64.

**Definition.** Let  $\text{At} \equiv \{p_0, p_1, p_2, \dots\} \cup \{\overline{p_0}, \overline{p_1}, \overline{p_2}, \dots\}$ . *Linear formulas* are the elements of the minimal set  $\text{Fm}$  such that

- $\text{At} \subset \text{Fm}$ ,
- if  $A \in \text{Fm}$  and  $B \in \text{Fm}$ , then  $(A \otimes B) \in \text{Fm}$  and  $(A \wp B) \in \text{Fm}$ .

$$\begin{array}{ll}
 (p_i)^\perp \equiv \overline{p_i} & (\overline{p_i})^\perp \equiv p_i \\
 (A \otimes B)^\perp \equiv (B)^\perp \wp (A)^\perp & (A \wp B)^\perp \equiv (B)^\perp \otimes (A)^\perp
 \end{array}$$

**Example.**  $((\overline{p} \wp ((\overline{r} \wp (\overline{r} \otimes r)) \otimes r)) \otimes q)^\perp = (\overline{q} \wp ((\overline{r} \wp ((\overline{r} \wp r) \otimes r)) \otimes p))$ .

**Definition.** The following function  $\tau: \text{Tp} \rightarrow \text{Fm}$  embeds  $L^*$  into cyclic linear logic.

$$\begin{array}{l}
 \tau(p_i) \equiv p_i \\
 \tau(A \cdot B) \equiv \tau(A) \otimes \tau(B) \\
 \tau(A \setminus B) \equiv \tau(A)^\perp \wp \tau(B) \\
 \tau(A/B) \equiv \tau(A) \wp \tau(B)^\perp
 \end{array}$$

**Example.**  $\tau(p_1/(p_2 \cdot p_3)) = p_1 \wp (\overline{p_3} \wp \overline{p_2})$

Derivable objects of cyclic linear logic are *sequents*  $\rightarrow A_1 \dots A_n$ , where  $A_i \in \text{Fm}$ .  
The intended meaning of  $\rightarrow A_1 \dots A_n$ , is  $A_1 \wp \dots \wp A_n$ .

**Axioms and rules**

$$\begin{array}{ccc} \rightarrow A^\perp A & \frac{\rightarrow \Gamma A B \Delta}{\rightarrow \Gamma (A \wp B) \Delta} (\wp) & \frac{\rightarrow \Gamma A \quad \rightarrow B \Delta}{\rightarrow \Gamma (A \otimes B) \Delta} (\otimes) \\ & \frac{\rightarrow \Gamma \Delta}{\rightarrow \Delta \Gamma} (\text{rotate}) & \frac{\rightarrow \Gamma A \quad \rightarrow A^\perp \Delta}{\rightarrow \Gamma \Delta} (\text{cut}) \end{array}$$

**Cut-elimination theorem.** We may drop (cut).

Another calculus for the same logic.

**Axioms and rules of MCLL**

$$\begin{array}{ccc} \frac{\rightarrow \Gamma A B \Delta}{\rightarrow \Gamma (A \wp B) \Delta} & \frac{\rightarrow \overline{p_i} p_i \quad \rightarrow p_i \overline{p_i}}{\rightarrow \Gamma A \quad \rightarrow \Phi B \Delta} & \frac{\rightarrow \Gamma A \Pi \quad \rightarrow B \Delta}{\rightarrow \Gamma (A \otimes B) \Delta \Pi} \\ & \frac{\rightarrow \Gamma A \quad \rightarrow \Phi B \Delta}{\rightarrow \Phi \Gamma (A \otimes B) \Delta} & \end{array}$$

**Example.**  $\text{MCLL} \vdash \rightarrow (\overline{p} \otimes q) (\overline{q} \otimes r) (\overline{r} \wp p)$ .

$$\frac{\frac{\frac{\rightarrow \overline{p} p \quad \rightarrow q \overline{q}}{\rightarrow (\overline{p} \otimes q) \overline{q} p} \quad \rightarrow r \overline{r}}{\rightarrow (\overline{p} \otimes q) (\overline{q} \otimes r) \overline{r} p}}{\rightarrow (\overline{p} \otimes q) (\overline{q} \otimes r) (\overline{r} \wp p)}$$

**Example.**  $\text{MCLL} \vdash \rightarrow (\overline{r} \otimes r) (\overline{r} \otimes r) (\overline{r} \wp r)$

**Remark.**  $L^* \vdash A_1 \dots A_n \rightarrow B$  if and only if  $\text{MCLL} \vdash \rightarrow \tau(A_n)^\perp \dots \tau(A_1)^\perp \tau(B)$ .

**Example.**  $L^* \vdash ((q \setminus r) \cdot s) \rightarrow (q \setminus (r \cdot s))$  and  $\text{MCLL} \vdash \rightarrow (\overline{s} \wp (\overline{r} \otimes q)) (\overline{q} \wp (r \otimes s))$ .

$$\frac{\frac{\frac{\frac{\rightarrow \overline{r} r \quad \rightarrow \overline{s} s}{\rightarrow \overline{s} \overline{r} (r \otimes s)} \quad \rightarrow q \overline{q}}{\rightarrow \overline{s} (\overline{r} \otimes q) \overline{q} (r \otimes s)}}{\rightarrow \overline{s} (\overline{r} \otimes q) (\overline{q} \wp (r \otimes s))}}{\rightarrow (\overline{s} \wp (\overline{r} \otimes q)) (\overline{q} \wp (r \otimes s))}$$

## 8 Complexity

M. Pentus, *Lambek calculus is NP-complete*, CUNY Ph.D. Program in Computer Science Technical Report TR-2003005, CUNY Graduate Center, New York, May 2003.

<http://www.cs.gc.cuny.edu/tr/techreport.php?id=79>

**Remark.** The derivability problem for MCLL is in NP.

**Theorem 6 (2003).** *The derivability problem for MCLL is NP-complete.*

We shall reformulate the well-known NP-complete problem *SAT* (satisfiability in the classical propositional logic) in terms of electrical circuits.

Let  $c_1 \wedge \dots \wedge c_m$  be a Boolean formula in conjunctive normal form with clauses  $c_1, \dots, c_m$  and variables  $x_1, \dots, x_n$ .

We construct a frame (with  $m$  lamps and  $n$  sockets) and a set of  $2n$  blocks (each of which fits into one socket only) so that the formula  $c_1 \wedge \dots \wedge c_m$  is satisfiable if and only if there is a way to plug  $n$  blocks into the sockets so that no lamp will be switched on. Each block (and each socket) has  $2m$  contacts.

**Example.**  $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_3)$ .

To model the circuits in MCLL we shall construct (in polynomial time) formulas  $G, E_i(0), E_i(1), F_i$  (where  $1 \leq i \leq n$ ) such that

- $c_1 \wedge \dots \wedge c_m$  is satisfiable if and only if  $\text{MCLL} \vdash \rightarrow E_1(t_1) \dots E_n(t_n) G$  for some  $t_1, \dots, t_n \in \{0, 1\}$ ,
- $\text{MCLL} \vdash \rightarrow F_1 \dots F_n G$  if and only if  $\text{MCLL} \vdash \rightarrow E_1(t_1) \dots E_n(t_n) G$  for some  $t_1, \dots, t_n \in \{0, 1\}$ .

We shall denote  $p_{n+1}$  by  $r$ .

In the following definitions  $1 \leq j < m$ ,  $1 \leq i \leq n$  and  $t \in \{0, 1\}$ .

$$\begin{aligned}
G^0 &\equiv (\bar{r} \wp r), \\
G^j &\equiv ((\bar{r} \wp G^{j-1}) \otimes r), \\
G &\equiv ((\bar{p}_n \wp G^{m-1}) \otimes p_0), \\
H^0 &\equiv (\bar{r} \otimes r), \\
H^j &\equiv ((\bar{r} \wp H^{j-1}) \otimes r), \\
H_i &\equiv ((\bar{p}_{i-1} \wp H^{m-1}) \otimes p_i), \\
E_i^0(t) &\equiv (\bar{r} \otimes r), \\
E_i^j(t) &\equiv \begin{cases} (\bar{r} \wp (E_i^{j-1}(t) \otimes r)) & \text{if } \llbracket x_i \rrbracket = t \rightarrow \llbracket c_j \rrbracket = 1, \\ ((\bar{r} \wp E_i^{j-1}(t)) \otimes r) & \text{otherwise,} \end{cases} \\
E_i(t) &\equiv \begin{cases} (\bar{p}_{i-1} \wp (E_i^{m-1}(t) \otimes p_i)) & \text{if } \llbracket x_i \rrbracket = t \rightarrow \llbracket c_m \rrbracket = 1, \\ ((\bar{p}_{i-1} \wp E_i^{m-1}(t)) \otimes p_i) & \text{otherwise,} \end{cases} \\
F_i &\equiv ((E_i(0) \otimes H_i^\perp) \wp H_i \wp (H_i^\perp \otimes E_i(1))).
\end{aligned}$$

**Lemma 2.**  $\text{MCLL} \vdash \rightarrow E_i(t) H_i^\perp$  for each  $1 \leq i \leq n$  and  $t \in \{0, 1\}$ .

**Lemma 3.**  $\text{MCLL} \vdash \rightarrow F_i E_i(t)^\perp$  for each  $1 \leq i \leq n$  and  $t \in \{0, 1\}$ .

**Lemma 4.** If  $\text{MCLL} \vdash \rightarrow \Gamma A^\perp$  and  $\text{MCLL} \vdash \rightarrow \Phi A \Delta$ , then  $\text{MCLL} \vdash \rightarrow \Phi \Gamma \Delta$ .

**Theorem 7 (2003).** The derivability problems for  $L^*$  and  $L$  are NP-complete.

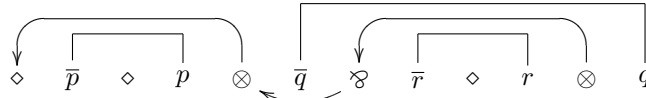
**Remark.** It is unknown whether the same holds for  $L(\setminus, /)^*$  and  $L(\setminus, /)$ .

## 9 Proof nets

**Example.** The derivation

$$\frac{\frac{\frac{\rightarrow \bar{r} r \quad \rightarrow \bar{q} q}{\rightarrow \bar{q} \bar{r} (r \otimes q)}}{\rightarrow \bar{p} p} \quad \rightarrow \bar{p} (p \otimes (\bar{q} \wp \bar{r})) (r \otimes q)}{\rightarrow \bar{p} (p \otimes (\bar{q} \wp \bar{r})) (r \otimes q)}$$

corresponds to the following proof net.



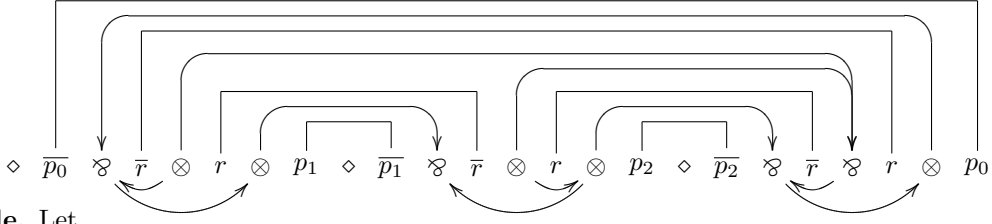
A proof net for  $\Gamma$  must satisfy the following conditions.

- $|\Gamma|_\wp + |\Gamma|_\otimes = |\Gamma|_\diamond + 2$ .
- No intersections.
- Acyclic.

**Example.** Let

$$\Gamma = ((\bar{p}_0 \wp (\bar{r} \otimes r)) \otimes p_1) (\bar{p}_1 \wp ((\bar{r} \otimes r) \otimes p_2)) ((\bar{p}_2 \wp (\bar{r} \wp r)) \otimes p_0).$$

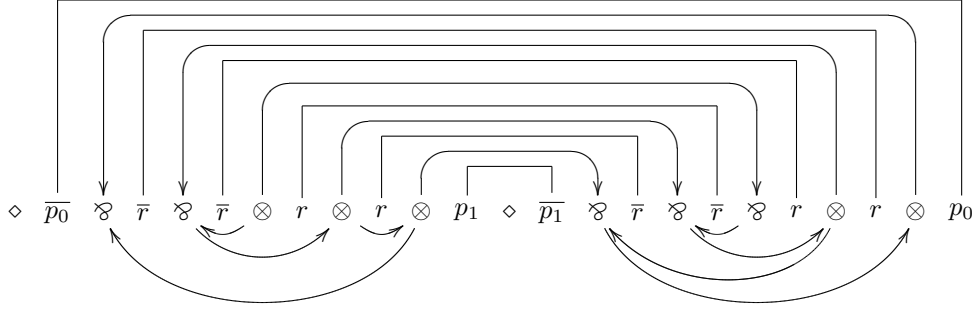
The following figure shows a proof net for  $\Gamma$ .



**Example.** Let

$$\Gamma = (\overline{p_0} \otimes (((\bar{r} \otimes (\bar{r} \otimes r)) \otimes r) \otimes p_1)) ((\overline{p_1} \otimes ((\bar{r} \otimes (\bar{r} \otimes r)) \otimes r)) \otimes p_0).$$

The following is not a valid proof net for  $\rightarrow \Gamma$  (it contains a cycle).



**Definition.**  $\|\cdot\|: \text{Fm} \rightarrow \mathbb{Z}$

$$\begin{aligned} \|p_i\| &= \|\overline{p_i}\| = 2, \\ \|A \otimes B\| &= \|A \otimes B\| = \|A\| + \|B\|, \\ \|A_1 \dots A_n\| &= \|A_1\| + \dots + \|A_n\|. \end{aligned}$$

**Definition.**  $\text{Occ} \Rightarrow \text{Fm} \times \mathbb{Z}$ .

**Definition.**  $c: \text{Occ} \rightarrow \mathbb{Z}$

$$\begin{aligned} c(p_i) &= c(\overline{p_i}) = 1, \\ c(A \otimes B) &= c(A \otimes B) = \|A\|. \end{aligned}$$

**Definition.**  $\prec$  is the following binary relation on  $\text{Occ}$ .

$$\begin{aligned} \langle A, k - \|A\| + c(A) \rangle &\prec \langle (A \lambda B), k \rangle, \\ \langle B, k + c(B) \rangle &\prec \langle (A \lambda B), k \rangle, \\ \text{if } \langle A, i \rangle &\prec \langle B, j \rangle \text{ and } \langle B, j \rangle \prec \langle C, k \rangle, \text{ then } \langle A, i \rangle \prec \langle C, k \rangle. \end{aligned}$$

Here  $\lambda \in \{\otimes, \otimes\}$ .

**Definition.** Let  $\diamond \notin \text{Fm}$ . Let  $\Gamma = A_1 \dots A_n$ . Then  $\Omega_\Gamma \Rightarrow \langle \Omega_\Gamma, \prec_\Gamma, <_\Gamma \rangle$ , where

$$\begin{aligned} \Omega_\Gamma &= \{ \langle B, k + \|A_1 \dots A_{i-1}\| \mid 1 \leq i \leq n \text{ and } \langle B, k \rangle \preceq \langle A_i, c(A_i) \rangle \} \\ &\cup \{ \langle \diamond, \|A_1 \dots A_{i-1}\| \mid 1 \leq i \leq n \}, \\ \langle A, k \rangle &\prec_\Gamma \langle B, l \rangle \text{ iff } A \neq \diamond, B \neq \diamond, \text{ and } \langle A, k \rangle \prec_\Gamma \langle B, l \rangle, \\ \langle A, k \rangle &<_\Gamma \langle B, l \rangle \text{ iff } k < l. \end{aligned}$$

**Definition.**

$$\begin{aligned} \Omega_\Gamma^\diamond &= \{ \langle C, k \rangle \in \Omega_\Gamma \mid C = \diamond \}, \\ \Omega_\Gamma^{\text{At}} &= \{ \langle C, k \rangle \in \Omega_\Gamma \mid C \in \text{At} \}, \\ \Omega_\Gamma^\otimes &= \{ \langle C, k \rangle \in \Omega_\Gamma \mid C = A \otimes B \text{ for some } A \text{ and } B \}, \\ \Omega_\Gamma^\otimes &= \{ \langle C, k \rangle \in \Omega_\Gamma \mid C = A \otimes B \text{ for some } A \text{ and } B \}. \end{aligned}$$

**Definition.** A *proof net* for  $\Gamma$  is a relational structure  $\langle \Omega_\Gamma, \mathcal{A}, \mathcal{E} \rangle$ , where

- $b(\Omega_\Gamma^\otimes) + b(\Omega_\Gamma^\circ) - b(\Omega_\Gamma^\otimes) = 2$ ,
- $\mathcal{A}$  is a map from  $\Omega_\Gamma^\otimes$  to  $\Omega_\Gamma^\otimes \cup \Omega_\Gamma^\circ$ ,
- $\mathcal{E}$  is a map from  $\Omega_\Gamma^{\text{At}}$  to  $\Omega_\Gamma^{\text{At}}$ ,
- if  $\langle \alpha, \beta \rangle \in \mathcal{E}$ , then  $\langle \beta, \alpha \rangle \in \mathcal{E}$ ,
- if  $\langle \langle A, i \rangle, \langle B, j \rangle \rangle \in \mathcal{E}$ , then  $A = B^\perp$ ,
- the edges of the graph  $\langle \Omega_\Gamma, \mathcal{A} \cup \mathcal{E} \rangle$  can be drawn without intersections on a semiplane while the vertices of the graph are ordered according to  $<_\Gamma$  on the border of the semiplane,
- the graph  $\langle \Omega_\Gamma, \prec_\Gamma \cup \mathcal{A} \rangle$  is acyclic.

**Theorem 8 (1998).**  $\text{MCLL} \vdash \rightarrow \Gamma$  if and only if there exists a proof net for  $\Gamma$ .

## 10 Equivalence

**Definition.**  $\text{MCLL} \vdash A \rightarrow B$  iff  $\text{MCLL} \vdash \rightarrow A^\perp B$ .

**Definition.**  $A \xleftrightarrow{\text{MCLL}} B$  iff  $\text{MCLL} \vdash A \rightarrow B$  and  $\text{MCLL} \vdash B \rightarrow A$ .

**Lemma 5.** •  $A \xleftrightarrow{\text{MCLL}} A$ .

- If  $A \xleftrightarrow{\text{MCLL}} B$ , then  $B \xleftrightarrow{\text{MCLL}} A$ .
- If  $A \xleftrightarrow{\text{MCLL}} B$  and  $B \xleftrightarrow{\text{MCLL}} C$ , then  $A \xleftrightarrow{\text{MCLL}} C$ .
- If  $A \xleftrightarrow{\text{MCLL}} B$  and  $C \xleftrightarrow{\text{MCLL}} D$ , then  $A \otimes C \xleftrightarrow{\text{MCLL}} B \otimes D$ .
- If  $A \xleftrightarrow{\text{MCLL}} B$  and  $C \xleftrightarrow{\text{MCLL}} D$ , then  $A \wp C \xleftrightarrow{\text{MCLL}} B \wp D$ .
- If  $A \xleftrightarrow{\text{MCLL}} B$ , then  $A^\perp \xleftrightarrow{\text{MCLL}} B^\perp$ .

**Definition.**  $\sharp: \text{Fm} \rightarrow \mathbb{Z}$

$$\begin{aligned} \sharp(p_i) &= \sharp(\bar{p}_i) = 0, \\ \sharp(A \wp B) &= \sharp A + \sharp B + 1, \\ \sharp(A \otimes B) &= \sharp A + \sharp B - 1. \end{aligned}$$

**Lemma 6.** If  $\text{MCLL} \vdash A \rightarrow B$ , then  $\sharp A = \sharp B$ .

**Definition.**  $\text{at}_0: \text{Fm} \rightarrow \mathcal{P}(\text{At})$  and  $\text{at}_1: \text{Fm} \rightarrow \mathcal{P}(\text{At})$ :

$$\begin{aligned} \text{at}_0(C) &= \{C\} \text{ if } C \in \text{At}, \\ \text{at}_1(C) &= \{C^\perp\} \text{ if } C \in \text{At}, \\ \text{at}_k(A \wp B) &= \text{at}_k(A \otimes B) = \text{at}_k(A) \cup \text{at}_{(k+1+\sharp A \bmod 2)}(B). \end{aligned}$$

**Lemma 7.** If  $A \xleftrightarrow{\text{MCLL}} B$ , then  $\text{at}_0(A) = \text{at}_0(B)$ .

**Theorem 9 (2002).**  $A \xleftrightarrow{\text{MCLL}} p_i$  if and only if  $\text{at}_0(A) = \{p_i\}$ ,  $\sharp A = 0$ , and  $\sharp C \in \{-1, 0, 1\}$  whenever  $C$  is a subformula of  $A$ .

**Corollary 2.** There is a deterministic polynomial time algorithm for the special equivalence problem: given  $A \in \text{Fm}$  and  $p_i$ , to decide whether  $A \xleftrightarrow{\text{MCLL}} p_i$ .

**Remark.** It is unknown whether the same holds for the problem  $A \xleftrightarrow{\text{MCLL}} B$ .