# Noncommutative linear logic

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## 1 Formal languages

A *formal language* is a set of finite words over a finite alphabet.

**Example.** Consider the alphabet $\Sigma = \{a, e, v\}$. The set $\{ve, veave, veaveave, veaveaveave, \ldots\}$ is a formal language.

Two important approaches to formal language specification:

- Noam Chomsky (recursion-theoretic approach)
- Jim Lambek (logico-algebraic approach)

  J. Lambek, *The mathematics of sentence structure*,
  American Mathematical Monthly **65** (1958), no. 3, 154–170.

By $\circ$ we denote the concatenation operator.
$\Sigma^*$ is the set of all words over the alphabet $\Sigma$.
$\Sigma^+$ is the set of all non-empty words over the alphabet $\Sigma$.

## 2 Lambek calculus

J. Lambek considers three basic operations on languages:

$$\begin{aligned} \mathcal{A} \cdot \mathcal{B} &\rightleftharpoons \{x \circ y \mid x \in \mathcal{A},\ y \in \mathcal{B}\}, \\ \mathcal{A} \backslash \mathcal{B} &\rightleftharpoons \{y \in \Sigma^+ \mid \mathcal{A} \cdot \{y\} \subseteq \mathcal{B}\}, \\ \mathcal{B} / \mathcal{A} &\rightleftharpoons \{x \in \Sigma^+ \mid \{x\} \cdot \mathcal{A} \subseteq \mathcal{B}\}. \end{aligned}$$

**Example.** Let $\mathcal{A} = \{j, m\}$ and $\mathcal{B} = \{je, jrj, jrm, me, mrj, mrm\}$.
Then $\mathcal{A} \backslash \mathcal{B} = \{e, rj, rm\}$.

**Definition.** *Types* are the elements of the minimal set Tp such that

- $\{p_0, p_1, p_2, \ldots\} \subset \mathrm{Tp}$
- If $A \in \mathrm{Tp}$ and $B \in \mathrm{Tp}$, then $(A \cdot B) \in \mathrm{Tp}$, $(A \backslash B) \in \mathrm{Tp}$, and $(A/B) \in \mathrm{Tp}$.

Derivable objects of $\mathrm{L_H}$ are $A \to B$, where $A \in \mathrm{Tp}$ and $B \in \mathrm{Tp}$.
**Axioms and rules of** $\mathrm{L_H}$

$$A \to A \qquad (A \cdot B) \cdot C \to A \cdot (B \cdot C) \qquad A \cdot (B \cdot C) \to (A \cdot B) \cdot C$$

$$\frac{A \to B \quad B \to C}{A \to C} \qquad \frac{A \cdot B \to C}{A \to C/B} \qquad \frac{A \cdot B \to C}{B \to A \backslash C}$$

$$\frac{A \to C/B}{A \cdot B \to C} \qquad \frac{B \to A \backslash C}{A \cdot B \to C}$$

We write $\mathrm{L_H} \vdash A \to B$ for "$A \to B$ is derivable in the calculus $\mathrm{L_H}$".

**Example.** Let $A, B \in \mathrm{Tp}$. Then $\mathrm{L_H} \vdash A \cdot (A\backslash B) \to B$.

$$\dfrac{A\backslash B \to A\backslash B}{A \cdot (A\backslash B) \to B}$$

**Remark.** There exist $A, B \in \mathrm{Tp}$ such that $\mathrm{L_H} \nvdash B \to A \cdot (A\backslash B)$.

**Example.** $A \cdot (B/C) \to (A \cdot B)/C$ is derivable in $\mathrm{L_H}$.

$$\dfrac{(A \cdot (B/C)) \cdot C \to A \cdot ((B/C) \cdot C) \qquad \dfrac{\dfrac{\dfrac{B/C \to B/C}{(B/C) \cdot C \to B} \quad \dfrac{A \cdot B \to A \cdot B}{B \to A\backslash(A \cdot B)}}{(B/C) \cdot C \to A\backslash(A \cdot B)}}{A \cdot ((B/C) \cdot C) \to A \cdot B}}{\dfrac{(A \cdot (B/C)) \cdot C \to A \cdot B}{A \cdot (B/C) \to (A \cdot B)/C}}$$

**Definition.** $A \underset{\mathrm{L_H}}{\leftrightarrow} B$ iff $\mathrm{L_H} \vdash A \to B$ and $\mathrm{L_H} \vdash B \to A$.

**Example.**

$$(A\backslash B)/C \underset{\mathrm{L_H}}{\leftrightarrow} A\backslash(B/C),$$
$$A/(B \cdot C) \underset{\mathrm{L_H}}{\leftrightarrow} (A/C)/B,$$
$$A \cdot (A\backslash(A \cdot B)) \underset{\mathrm{L_H}}{\leftrightarrow} A \cdot B.$$

**Example.**

$$\mathrm{L_H} \vdash ((B/A)\backslash C)\backslash D \to (B\backslash C)\backslash(A\backslash D),$$
$$\mathrm{L_H} \nvdash ((A\backslash B)\backslash C)\backslash D \to C\backslash((B\backslash A)\backslash D).$$

# 3 Lambek calculus L with sequents

Derivable objects of the calculus L are *sequents* $\Gamma \to A$, where $A \in \mathrm{Tp}$ and $\Gamma \in \mathrm{Tp}^+$.

**Axioms and rules of** L

$$A \to A$$

$$\dfrac{\Phi \to B \quad \Gamma\, B\, \Delta \to A}{\Gamma\, \Phi\, \Delta \to A}\ (\text{cut})$$

$$\dfrac{A\, \Pi \to B}{\Pi \to A\backslash B}\ (\to \backslash),\ \text{where } \Pi \neq \Lambda$$

$$\dfrac{\Phi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, \Phi\, (A\backslash B)\, \Delta \to C}\ (\backslash \to)$$

$$\dfrac{\Pi\, A \to B}{\Pi \to B/A}\ (\to /),\ \text{where } \Pi \neq \Lambda$$

$$\dfrac{\Phi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, (B/A)\, \Phi\, \Delta \to C}\ (/ \to)$$

$$\dfrac{\Gamma \to A \quad \Delta \to B}{\Gamma\, \Delta \to A \cdot B}\ (\to \cdot)$$

$$\dfrac{\Gamma\, A\, B\, \Delta \to C}{\Gamma\, (A \cdot B)\, \Delta \to C}\ (\cdot \to)$$

Here $\Lambda$ is the empty sequence, $A, B, C \in \mathrm{Tp}$, and $\Gamma, \Delta, \Phi, \Pi \in \mathrm{Tp}^*$.

**Theorem 1 (J. Lambek, 1958).** $\mathrm{L} \vdash A_1 \ldots A_n \to B$ *if and only if* $\mathrm{L_H} \vdash A_1 \cdot \ldots \cdot A_n \to B$.

**Cut-elimination theorem (J. Lambek, 1958).** A sequent is derivable in L if and only if it is derivable in L without (cut).

**Example.** $\mathrm{L} \vdash A \cdot (B/C) \to (A \cdot B)/C$

$$\dfrac{\dfrac{\dfrac{A \to A \qquad \dfrac{C \to C \quad B \to B}{(B/C)\, C \to B}\ (/ \to)}{A\, (B/C)\, C \to (A \cdot B)}\ (\to \cdot)}{A\, (B/C) \to (A \cdot B)/C}\ (\to /)}{A \cdot (B/C) \to (A \cdot B)/C}\ (\cdot \to)$$

**Remark.** $\mathrm{L} \nvdash (A \cdot B)/C \to A \cdot (B/C)$.

# 4 Grammars

**Definition.** A *Lambek categorial grammar* is a triple $\langle \Sigma, D, f \rangle$ such that $|\Sigma| < \infty$, $D \in \mathrm{Tp}$, $f \colon \Sigma \to \mathcal{P}(\mathrm{Tp})$, and $|f(t)| < \infty$ for each $t \in \Sigma$.
The grammar *recognizes* the language

$$\begin{array}{c} \mathcal{L}_L(\Sigma, D, f) \rightleftharpoons \{t_1 \ldots t_n \in \Sigma^+ \mid \exists B_1 \in f(t_1) \ldots \exists B_n \in f(t_n) \\ \mathrm{L} \vdash B_1 \ldots B_n \to D\} \end{array}$$

**Example.**

$$np = p_1 \quad s = p_2 \quad D = s \quad \Sigma = \{\text{John}, \text{Mary}, \text{works}, \text{recommends}\}$$

$$\begin{aligned} f(\text{John}) = f(\text{Mary}) &= \{np\} \\ f(\text{works}) &= \{(np\backslash s)\} \\ f(\text{recommends}) &= \{((np\backslash s)/np)\} \end{aligned}$$

$$\dfrac{np \to np \qquad \dfrac{np \to np \quad s \to s}{np\,(np\backslash s) \to s}\;(\backslash \to)}{\begin{array}{cccc} np & ((np\backslash s)/np) & np & \to s \\ \text{John} & \text{recommends} & \text{Mary} & \end{array}}\;(/ \to)$$

B. Carpenter, *Type-Logical Semantics*, MIT Press, Cambridge, MA, 1997.
`http://www.colloquial.com/tlg/parser.html`

**Example.**

$$\Sigma = \{\text{Val}, \text{recommends}, \text{he}, \text{she}, \text{him}, \text{her}\}$$

$$\begin{aligned} f(\text{Val}) &= \{np\} \\ f(\text{recommends}) &= \{((np\backslash s)/np)\} \\ f(\text{he}) = f(\text{she}) &= \{(s/(np\backslash s))\} \\ f(\text{him}) = f(\text{her}) &= \{((s/np)\backslash s)\} \end{aligned}$$

$$\dfrac{\dfrac{\dfrac{np \to np \qquad \dfrac{(np\backslash s) \to (np\backslash s) \quad s \to s}{(s/(np\backslash s))\;(np\backslash s) \to s}\;(/ \to)}{(s/(np\backslash s))\;((np\backslash s)/np)\;np \to s}\;(/ \to)}{(s/(np\backslash s))\;((np\backslash s)/np) \to (s/np)}\;(\to /) \qquad s \to s}{\begin{array}{cccc} (s/(np\backslash s)) & ((np\backslash s)/np) & ((s/np)\backslash s) & \to s \\ \text{She} & \text{recommends} & \text{him} & \end{array}}\;(\backslash \to)$$

**Example.**

$$\begin{aligned} \Sigma = \{&\text{John}, \text{Val}, \text{succeeds}, \text{exists}, \text{helps}, \text{recommends}, \\ &\text{student}, \text{professor}, \text{club}, \text{a}, \text{the}, \text{every}, \text{this}, \text{strange}, \\ &\text{whenever}, \text{whom}, \text{relatively}, \text{everywhere}, \text{or}\} \end{aligned}$$

John succeeds whenever Val recommends a club or helps
the student whom this relatively strange professor recommends.

$$\begin{aligned} f(\text{Val}) &= \{np\} \\ f(\text{succeeds}) = f(\text{exists}) &= \{(np\backslash s)\} \\ f(\text{helps}) = f(\text{recommends}) &= \{((np\backslash s)/np)\} \\ f(\text{student}) = f(\text{professor}) = f(\text{club}) &= \{n\} \\ f(\text{a}) = f(\text{the}) = f(\text{every}) &= \{(np/n)\} \\ f(\text{this}) &= \{(np/n), np\} \\ f(\text{strange}) &= \{(n/n)\} \\ f(\text{whenever}) &= \{((s\backslash s)/s)\} \\ f(\text{whom}) &= \{((n\backslash n)/(s/np))\} \\ f(\text{relatively}) &= \{((n/n)/(n/n))\} \\ f(\text{everywhere}) &= \{((np\backslash s)\backslash(np\backslash s))\} \end{aligned}$$

$$f(\text{or}) = \{((np\backslash np)/np), ((s\backslash s)/s), (((np\backslash s)\backslash(np\backslash s))/(np\backslash s))\}$$

**Definition.** A *context-free grammar* is a 4-tuple $\langle \Sigma, \mathcal{W}, S, \mathcal{R} \rangle$ such that $|\Sigma| < \infty$, $|\mathcal{W}| < \infty$, $\Sigma \cap \mathcal{W} = \varnothing$, $S \in \mathcal{W}$,
$\mathcal{R} \subset \{A \mapsto u \mid A \in \mathcal{W} \text{ and } u \in (\Sigma \cup \mathcal{W})^+\}$, and $|\mathcal{R}| < \infty$.
The grammar *recognizes* the language

$$\mathcal{G}(\Sigma, \mathcal{W}, S, \mathcal{R}) \rightleftharpoons \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R}) \cap \Sigma^+.$$

Here $\bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$ is defined inductively.

- $S \in \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$
- If $u_1, u_2, u_3 \in (\Sigma \cup \mathcal{W})^*$, $A \in \mathcal{W}$, $u_1 A u_3 \in \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$, and $A \mapsto u_2 \in \mathcal{R}$, then $u_1 u_2 u_3 \in \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$.

**Example.**

$$\Sigma = \{\text{John}, \text{Mary}, \text{works}, \text{recommends}\} \qquad \mathcal{W} = \{S, NP, VP, V_t\}$$

$$\mathcal{R} = \{S \mapsto NP \;\; VP, \quad VP \mapsto V_t \; NP, \quad NP \mapsto \text{John}, \quad NP \mapsto \text{Mary}, \quad VP \mapsto \text{works}, \quad V_t \mapsto \text{recommends}\}$$

**Theorem 2 (J. M. Cohen, 1967).**

$$\forall \langle \Sigma, \mathcal{W}, S, \mathcal{R} \rangle \; \exists D \; \exists f \; \textit{such that } \mathcal{L}_L(\Sigma, D, f) = \mathcal{G}(\Sigma, \mathcal{W}, S, \mathcal{R})$$

**Theorem 3 (1992).**

$$\forall \langle \Sigma, D, f \rangle \; \exists \mathcal{W} \; \exists S \; \exists \mathcal{R} \; \textit{such that } \mathcal{G}(\Sigma, \mathcal{W}, S, \mathcal{R}) = \mathcal{L}_L(\Sigma, D, f)$$

**Definition.**

$$\|p_i\| \rightleftharpoons 1,$$
$$\|A \cdot B\| = \|A \backslash B\| = \|A / B\| \rightleftharpoons \|A\| + \|B\|.$$

Proof of Theorem 3.

$$m \rightleftharpoons \max(\|D\|, \max_{t \in \Sigma} \max_{B \in f(t)} \|B\|)$$

$$\begin{aligned}
\mathcal{W} &\rightleftharpoons \{A \in \text{Tp} \mid \|A\| \leq m\} \\
S &\rightleftharpoons D \\
\mathcal{R} &\rightleftharpoons \{B \mapsto t \mid t \in \Sigma \text{ and } B \in f(t)\} \cup \\
&\cup \{C \mapsto AB \mid A, B, C \in \mathcal{W} \text{ and } \mathrm{L} \vdash AB \to C\} \cup \\
&\cup \{D \mapsto A \mid A \in \mathcal{W} \text{ and } \mathrm{L} \vdash A \to D\}
\end{aligned}$$

□

**Example.**

$$\Sigma = \{\text{John}, \text{Mary}, \text{recommends}\}$$

$$\begin{aligned}
np &\mapsto \text{John} \in \mathcal{R} \\
np &\mapsto \text{Mary} \in \mathcal{R} \\
((np \backslash s)/np) &\mapsto \text{recommends} \in \mathcal{R} \\
s &\mapsto np \;\; (np \backslash s) \in \mathcal{R} \\
(np \backslash s) &\mapsto ((np \backslash s)/np) \;\; np \in \mathcal{R} \\
&\text{etc.}
\end{aligned}$$

Theorem 3 follows from Lemma 1.

**Lemma 1.** *If* $\mathrm{L} \vdash B_1 \ldots B_n \to D$, *where* $n \geq 2$, $\|D\| \leq m$, *and* $\|B_i\| \leq m$ *for each* $i$, *then* $B_1 \ldots B_n \to D$ *follows by means of the cut rule from* $n-1$ *derivable sequents of the form* $A_1 A_2 \to A_3$, *where* $\|A_j\| \leq m$ *for each* $j$.

## 5 Language models

**Definition.** A *language model* (*free semigroup model*) is a pair $\langle \Sigma^+, v\rangle$ such that $\Sigma$ is a finite or countable alphabet and

- $v(p_i) \subseteq \Sigma^+$,
- $v(A \cdot B) = v(A) \cdot v(B)$,
- $v(A\backslash B) = v(A)\backslash v(B) = \{y \in \Sigma^+ \mid v(A) \cdot \{y\} \subseteq v(B)\}$,
- $v(B/A) = v(B)/v(A) = \{x \in \Sigma^+ \mid \{x\} \cdot v(A) \subseteq v(B)\}$.

**Remark.** L is sound with respect to language models.

**Definition.** L$(\backslash, /)$ is the elementary fragment of L without $\cdot$.

**Remark.** L is conservative over L$(\backslash, /)$.

**Remark (W. Buszkowski, 1982).** L$(\backslash, /)$ is complete with respect to language models.

Proof.

$$\Sigma \rightleftharpoons \mathrm{Tp}$$
$$v(A) \rightleftharpoons \{\Gamma \in \mathrm{Tp}^+ \mid \mathrm{L} \vdash \Gamma \to A\}$$

□

**Theorem 4 (1993).** *A sequent is derivable in* L *if and only if it is true in every language model.*

**Example.** Let $p, q \in \mathrm{Pr}$. Then $\mathrm{L} \nvdash p \to p \cdot (q\backslash q)$.

$$\Sigma = \{a_1, a_2\} \qquad v(p) = \{a_1\}$$
$$v(q) = \{a_2\}$$

$$v(q\backslash q) = \varnothing$$
$$v(p \cdot (q\backslash q)) = \varnothing$$

$$v(p) = \{a_1\} \nsubseteq \varnothing = v(p \cdot (q\backslash q))$$

**Example.** Let $p, q, r \in \mathrm{Pr}$. Then $\mathrm{L} \nvdash (p \cdot q)/r \to p \cdot (q/r)$.

$$\Sigma = \{a_1, a_2, a_3\} \qquad v(p) = \{a_1a_2\}$$
$$v(q) = \{a_3\}$$
$$v(r) = \{a_2a_3\}$$

$$v(p \cdot q) = \{a_1a_2a_3\}$$
$$v((p \cdot q)/r) = \{a_1\}$$
$$v(q/r) = \varnothing$$
$$v(p \cdot (q/r)) = \varnothing$$

$$v((p \cdot q)/r) = \{a_1\} \nsubseteq \varnothing = v(p \cdot (q/r))$$

**Example.**

$$\Sigma' = \{b, c\} \qquad v'(p) = \{bcbbccb\}$$
$$v'(q) = \{bcccb\}$$
$$v'(r) = \{bccbbcccb\}$$

**Corollary 1.** *A sequent is derivable in* L *if and only if it is true in every language model over a two-symbol alphabet.*

Proof. Let $\Sigma = \{a_1, a_2, \ldots\}$. Put $\Sigma' = \{b, c\}$.
Map $a_i$ to $b\underbrace{cc\ldots c}_{i}b$. □

## 6 The calculus L*

Derivable objects of the calculus $\mathrm{L}^*$ are *sequents* $\Gamma \to A$, where $A \in \mathrm{Tp}$ and $\Gamma \in \mathrm{Tp}^*$.

**Axioms and rules of** $\mathrm{L}^*$

$$A \to A \qquad \frac{\Phi \to B \quad \Gamma\, B\, \Delta \to A}{\Gamma\, \Phi\, \Delta \to A}\ (\text{cut})$$

$$\frac{A\, \Pi \to B}{\Pi \to A \backslash B}\ (\to \backslash) \qquad \frac{\Phi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, \Phi\, (A \backslash B)\, \Delta \to C}\ (\backslash \to)$$

$$\frac{\Pi\, A \to B}{\Pi \to B / A}\ (\to /) \qquad \frac{\Phi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, (B/A)\, \Phi\, \Delta \to C}\ (/ \to)$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma\, \Delta \to A \cdot B}\ (\to \cdot) \qquad \frac{\Gamma\, A\, B\, \Delta \to C}{\Gamma\, (A \cdot B)\, \Delta \to C}\ (\cdot \to)$$

**Example.**

$$\frac{A \to A \quad \dfrac{B \to B}{\to B \backslash B}\ (\to \backslash)}{A \to A \cdot (B \backslash B)}\ (\to \cdot)$$

**Remark.** $\mathrm{L}^* \vdash A \to A \cdot (B \backslash B)$, but $\mathrm{L} \nvdash A \to A \cdot (B \backslash B)$.

**Cut-elimination theorem.** We may drop (cut).

**Definition.** A *free monoid model* is a pair $\langle \Sigma^*, v \rangle$ such that $\Sigma$ is a finite or countable alphabet and

- $v(p_i) \subseteq \Sigma^*$,
- $v(A \cdot B) = v(A) \cdot v(B)$,
- $v(A \backslash B) = \{y \in \Sigma^* \mid v(A) \cdot \{y\} \subseteq v(B)\}$,
- $v(B/A) = \{x \in \Sigma^* \mid \{x\} \cdot v(A) \subseteq v(B)\}$.

**Theorem 5 (1996).** *A sequent is derivable in* $\mathrm{L}^*$ *if and only if it is true in every free monoid model.*

## 7 Cyclic linear logic MCLL

We consider only multiplicative fragments of linear logic calculi.

D. N. Yetter, *Quantales and noncommutative linear logic*, Journal of Symbolic Logic, **55** (1990), no. 1, pp. 41–64.

**Definition.** Let $\mathrm{At} \rightleftharpoons \{p_0, p_1, p_2, \ldots\} \cup \{\overline{p_0}, \overline{p_1}, \overline{p_2}, \ldots\}$. *Linear formulas* are the elements of the minimal set Fm such that

- $\mathrm{At} \subset \mathrm{Fm}$,
- if $A \in \mathrm{Fm}$ and $B \in \mathrm{Fm}$, then $(A \otimes B) \in \mathrm{Fm}$ and $(A \invamp B) \in \mathrm{Fm}$.

$$(p_i)^\perp \rightleftharpoons \overline{p_i} \qquad (\overline{p_i})^\perp \rightleftharpoons p_i$$

$$(A \otimes B)^\perp \rightleftharpoons (B)^\perp \invamp (A)^\perp \qquad (A \invamp B)^\perp \rightleftharpoons (B)^\perp \otimes (A)^\perp$$

**Example.** $((\overline{p} \invamp ((\overline{r} \invamp (\overline{r} \otimes r)) \otimes r)) \otimes q)^\perp = (\overline{q} \invamp ((\overline{r} \invamp ((\overline{r} \invamp r) \otimes r)) \otimes p))$.

**Definition.** The following function $\tau \colon \mathrm{Tp} \to \mathrm{Fm}$ embeds $\mathrm{L}^*$ into cyclic linear logic.

$$\begin{aligned} \tau(p_i) &\rightleftharpoons p_i \\ \tau(A \cdot B) &\rightleftharpoons \tau(A) \otimes \tau(B) \\ \tau(A \backslash B) &\rightleftharpoons \tau(A)^\perp \invamp \tau(B) \\ \tau(A/B) &\rightleftharpoons \tau(A) \invamp \tau(B)^\perp \end{aligned}$$

**Example.** $\tau(p_1/(p_2 \cdot p_3)) = p_1 \parr (\overline{p_3} \parr \overline{p_2})$

Derivable objects of cyclic linear logic are *sequents* $\to A_1 \ \ldots\ A_n$, where $A_i \in \mathrm{Fm}$.
The intended meaning of $\to A_1 \ \ldots\ A_n$, is $A_1 \parr \ldots \parr A_n$.
**Axioms and rules**

$$\to A^{\perp}\, A \qquad \frac{\to \Gamma\, A\, B\, \Delta}{\to \Gamma\, (A \parr B)\, \Delta}\,(\parr) \qquad \frac{\to \Gamma\, A \quad \to B\, \Delta}{\to \Gamma\, (A \otimes B)\, \Delta}\,(\otimes)$$

$$\frac{\to \Gamma\, \Delta}{\to \Delta\, \Gamma}\,(\text{rotate}) \qquad \frac{\to \Gamma\, A \quad \to A^{\perp}\, \Delta}{\to \Gamma\, \Delta}\,(\text{cut})$$

**Cut-elimination theorem.** We may drop (cut).

Another calculus for the same logic.
**Axioms and rules of** MCLL

$$\to \overline{p_i}\, p_i \qquad \to p_i\, \overline{p_i}$$

$$\frac{\to \Gamma\, A\, B\, \Delta}{\to \Gamma\, (A \parr B)\, \Delta} \qquad \frac{\to \Gamma\, A \quad \to \Phi\, B\, \Delta}{\to \Phi\, \Gamma\, (A \otimes B)\, \Delta} \qquad \frac{\to \Gamma\, A\, \Pi \quad \to B\, \Delta}{\to \Gamma\, (A \otimes B)\, \Delta\, \Pi}$$

**Example.** $\mathrm{MCLL} \vdash\ \to (\overline{p} \otimes q)\, (\overline{q} \otimes r)\, (\overline{r} \parr p)$.

$$\cfrac{\cfrac{\cfrac{\to \overline{p}\, p \quad \to q\, \overline{q}}{\to (\overline{p} \otimes q)\, \overline{q}\, p} \quad \to r\, \overline{r}}{\to (\overline{p} \otimes q)\, (\overline{q} \otimes r)\, \overline{r}\, p}}{\to (\overline{p} \otimes q)\, (\overline{q} \otimes r)\, (\overline{r} \parr p)}$$

**Example.** $\mathrm{MCLL} \vdash\ \to (\overline{r} \otimes r)\, (\overline{r} \otimes r)\, (\overline{r} \parr r)$

**Remark.** $\mathrm{L}^* \vdash A_1 \ \ldots\ A_n \to B$ if and only if $\mathrm{MCLL} \vdash\ \to \tau(A_n)^{\perp} \ \ldots\ \tau(A_1)^{\perp}\, \tau(B)$.

**Example.** $\mathrm{L}^* \vdash ((q\backslash r) \cdot s) \to (q\backslash (r \cdot s))$ and $\mathrm{MCLL} \vdash\ \to (\overline{s} \parr (\overline{r} \otimes q))\, (\overline{q} \parr (r \otimes s))$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\to \overline{r}\, r \quad \to \overline{s}\, s}{\to \overline{s}\, \overline{r}\, (r \otimes s)} \quad \to q\, \overline{q}}{\to \overline{s}\, (\overline{r} \otimes q)\, \overline{q}\, (r \otimes s)}}{\to \overline{s}\, (\overline{r} \otimes q)\, (\overline{q} \parr (r \otimes s))}}{\to (\overline{s} \parr (\overline{r} \otimes q))\, (\overline{q} \parr (r \otimes s))}$$

# 8 Complexity

M. Pentus, *Lambek calculus is* NP*-complete*, CUNY Ph.D. Program in Computer Science Technical Report TR-2003005, CUNY Graduate Center, New York, May 2003.
`http://www.cs.gc.cuny.edu/tr/techreport.php?id=79`

**Remark.** The derivability problem for MCLL is in NP.

**Theorem 6 (2003).** *The derivability problem for* MCLL *is* NP*-complete.*

We shall reformulate the well-known NP-complete problem *SAT* (satisfiability in the classical propositional logic) in terms of electrical circuits.

Let $c_1 \wedge \ldots \wedge c_m$ be a Boolean formula in conjunctive normal form with clauses $c_1, \ldots, c_m$ and variables $x_1, \ldots, x_n$.

We construct a frame (with $m$ lamps and $n$ sockets) and a set of $2n$ blocks (each of which fits into one socket only) so that the formula $c_1 \wedge \ldots \wedge c_m$ is satisfiable if and only if there is a way to plug $n$ blocks into the sockets so that no lamp will be switched on. Each block (and each socket) has $2m$ contacts.

**Example.** $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_3)$.

To model the circuits in MCLL we shall construct (in polynomial time) formulas $G$, $E_i(0)$, $E_i(1)$, $F_i$ (where $1 \le i \le n$) such that

- $c_1 \wedge \ldots \wedge c_m$ is satisfiable if and only if $\mathrm{MCLL} \vdash\ \to E_1(t_1)\ \ldots\ E_n(t_n)\, G$ for some $t_1, \ldots, t_n \in \{0,1\}$,
- $\mathrm{MCLL} \vdash\ \to F_1\ \ldots\ F_n\, G$ if and only if $\mathrm{MCLL} \vdash\ \to E_1(t_1)\ \ldots\ E_n(t_n)\, G$ for some $t_1, \ldots, t_n \in \{0,1\}$.

We shall denote $p_{n+1}$ by $r$.
In the following definitions $1 \le j < m$, $1 \le i \le n$ and $t \in \{0,1\}$.

$$\begin{aligned}
G^0 &\rightleftharpoons (\overline{r} \parr r),\\
G^j &\rightleftharpoons ((\overline{r} \parr G^{j-1}) \otimes r),\\
G &\rightleftharpoons ((\overline{p_n} \parr G^{m-1}) \otimes p_0),\\
H^0 &\rightleftharpoons (\overline{r} \otimes r),\\
H^j &\rightleftharpoons ((\overline{r} \parr H^{j-1}) \otimes r),\\
H_i &\rightleftharpoons ((\overline{p_{i-1}} \parr H^{m-1}) \otimes p_i),\\
E_i^0(t) &\rightleftharpoons (\overline{r} \otimes r),\\
E_i^j(t) &\rightleftharpoons \begin{cases} (\overline{r} \parr (E_i^{j-1}(t) \otimes r)) & \text{if } [\![x_i]\!] = t \to [\![c_j]\!] = 1,\\ ((\overline{r} \parr E_i^{j-1}(t)) \otimes r) & \text{otherwise,} \end{cases}\\
E_i(t) &\rightleftharpoons \begin{cases} (\overline{p_{i-1}} \parr (E_i^{m-1}(t) \otimes p_i)) & \text{if } [\![x_i]\!] = t \to [\![c_m]\!] = 1,\\ ((\overline{p_{i-1}} \parr E_i^{m-1}(t)) \otimes p_i) & \text{otherwise,} \end{cases}\\
F_i &\rightleftharpoons ((E_i(0) \otimes H_i^{\perp}) \parr H_i \parr (H_i^{\perp} \otimes E_i(1))).
\end{aligned}$$

**Lemma 2.** $\mathrm{MCLL} \vdash\ \to E_i(t)\, H_i^{\perp}$ *for each* $1 \le i \le n$ *and* $t \in \{0,1\}$.

**Lemma 3.** $\mathrm{MCLL} \vdash\ \to F_i\, E_i(t)^{\perp}$ *for each* $1 \le i \le n$ *and* $t \in \{0,1\}$.

**Lemma 4.** *If* $\mathrm{MCLL} \vdash\ \to \Gamma\, A^{\perp}$ *and* $\mathrm{MCLL} \vdash\ \to \Phi\, A\, \Delta$*, then* $\mathrm{MCLL} \vdash\ \to \Phi\, \Gamma\, \Delta$.

**Theorem 7 (2003).** *The derivability problems for* $\mathrm{L}^*$ *and* $\mathrm{L}$ *are* NP*-complete.*

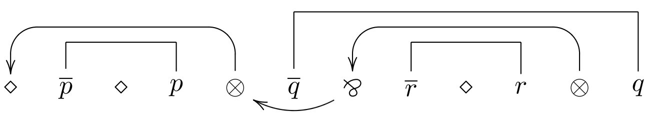
**Remark.** It is unknown whether the same holds for $\mathrm{L}(\backslash, /)^*$ and $\mathrm{L}(\backslash, /)$.

# 9 Proof nets

**Example.** The derivation

$$\dfrac{\to \overline{p}\,p \qquad \dfrac{\to \overline{r}\,r \qquad \to \overline{q}\,q}{\to \overline{q}\,\overline{r}\,(r \otimes q)}}{\to \overline{p}\,(p \otimes (\overline{q} \parr \overline{r}))\,(r \otimes q)}$$

corresponds to the following proof net.



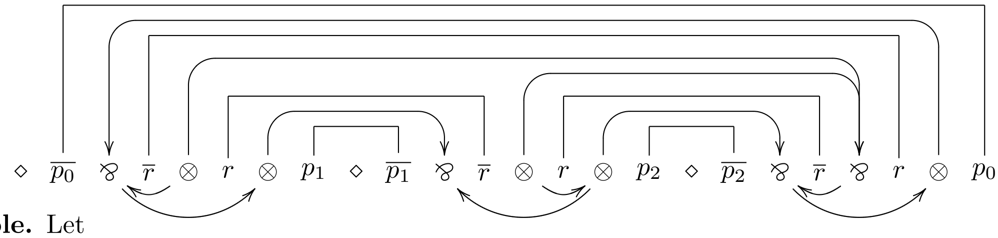

A *proof net* for $\Gamma$ must satisfy the following conditions.

- $|\Gamma|_{\parr} + |\Gamma|_{\diamond} = |\Gamma|_{\otimes} + 2$.
- No intersections.
- Acyclic.

**Example.** Let

$$\Gamma = ((\overline{p_0} \parr (\overline{r} \otimes r)) \otimes p_1)\, (\overline{p_1} \parr ((\overline{r} \otimes r) \otimes p_2))\, ((\overline{p_2} \parr (\overline{r} \parr r)) \otimes p_0).$$

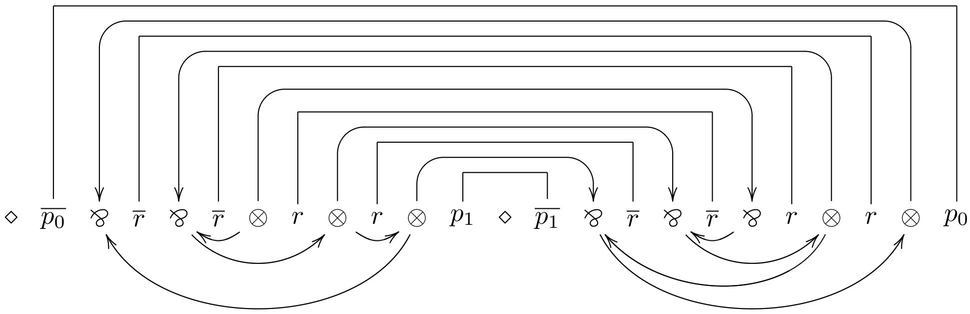
The following figure shows a proof net for $\Gamma$.



**Example.** Let

$$\Gamma = (\overline{p_0} \parr (((\overline{r} \parr (\overline{r} \otimes r)) \otimes r) \otimes p_1))\,((\overline{p_1} \parr ((\overline{r} \parr (\overline{r} \parr r)) \otimes r)) \otimes p_0).$$

The following is not a valid proof net for $\to \Gamma$ (it contains a cycle).



**Definition.** $\|\cdot\|\colon \mathrm{Fm} \to \mathbb{Z}$

$$\begin{aligned}
&\|p_i\| = \|\overline{p_i}\| \rightleftharpoons 2,\\
&\|A \otimes B\| = \|A \parr B\| \rightleftharpoons \|A\| + \|B\|,\\
&\|A_1 \dots A_n\| \rightleftharpoons \|A_1\| + \dots + \|A_n\|.
\end{aligned}$$

**Definition.** $\mathrm{Occ} \rightleftharpoons \mathrm{Fm} \times \mathbb{Z}$.

**Definition.** $c\colon \mathrm{Occ} \to \mathbb{Z}$

$$\begin{aligned}
&c(p_i) = c(\overline{p_i}) \rightleftharpoons 1,\\
&c(A \otimes B) = c(A \parr B) \rightleftharpoons \|A\|.
\end{aligned}$$

**Definition.** $\prec$ is the following binary relation on Occ.

$$\begin{aligned}
&\langle A, k - \|A\| + c(A)\rangle \prec \langle (A \mathbin{\lambda} B), k\rangle,\\
&\langle B, k + c(B)\rangle \prec \langle (A \mathbin{\lambda} B), k\rangle,\\
&\text{if } \langle A, i\rangle \prec \langle B, j\rangle \text{ and } \langle B, j\rangle \prec \langle C, k\rangle, \text{ then } \langle A, i\rangle \prec \langle C, k\rangle.
\end{aligned}$$

Here $\lambda \in \{\otimes, \parr\}$.

**Definition.** Let $\diamond \notin \mathrm{Fm}$. Let $\Gamma = A_1 \dots A_n$. Then $\mathbf{\Omega}_\Gamma \rightleftharpoons \langle \Omega_\Gamma, \prec_\Gamma, <_\Gamma\rangle$, where

$$\begin{aligned}
&\Omega_\Gamma \rightleftharpoons \{\langle B, k + \|A_1 \dots A_{i-1}\|\rangle \mid 1 \le i \le n \text{ and } \langle B, k\rangle \preceq \langle A_i, c(A_i)\rangle\}\\
&\quad \cup \{\langle \diamond, \|A_1 \dots A_{i-1}\|\rangle \mid 1 \le i \le n\},\\
&\langle A, k\rangle \prec_\Gamma \langle B, l\rangle \text{ iff } A \ne \diamond,\ B \ne \diamond, \text{ and } \langle A, k\rangle \prec_\Gamma \langle B, l\rangle,\\
&\langle A, k\rangle <_\Gamma \langle B, l\rangle \text{ iff } k < l.
\end{aligned}$$

**Definition.**

$$\begin{aligned}
\Omega_\Gamma^\diamond &\rightleftharpoons \{\langle C, k\rangle \in \Omega_\Gamma \mid C = \diamond\},\\
\Omega_\Gamma^{\mathrm{At}} &\rightleftharpoons \{\langle C, k\rangle \in \Omega_\Gamma \mid C \in \mathrm{At}\},\\
\Omega_\Gamma^{\otimes} &\rightleftharpoons \{\langle C, k\rangle \in \Omega_\Gamma \mid C = A \otimes B \text{ for some } A \text{ and } B\},\\
\Omega_\Gamma^{\parr} &\rightleftharpoons \{\langle C, k\rangle \in \Omega_\Gamma \mid C = A \parr B \text{ for some } A \text{ and } B\}.
\end{aligned}$$

**Definition.** A *proof net for* $\Gamma$ is a relational structure $\langle \mathbf{\Omega}_\Gamma, \mathcal{A}, \mathcal{E}\rangle$, where

- $\flat(\Omega_\Gamma^{⅋}) + \flat(\Omega_\Gamma^{\diamond}) - \flat(\Omega_\Gamma^{\otimes}) = 2$,
- $\mathcal{A}$ is a map from $\Omega_\Gamma^{\otimes}$ to $\Omega_\Gamma^{⅋} \cup \Omega_\Gamma^{\diamond}$,
- $\mathcal{E}$ is a map from $\Omega_\Gamma^{\mathrm{At}}$ to $\Omega_\Gamma^{\mathrm{At}}$,
- if $\langle \alpha, \beta\rangle \in \mathcal{E}$, then $\langle \beta, \alpha\rangle \in \mathcal{E}$,
- if $\langle\langle A, i\rangle, \langle B, j\rangle\rangle \in \mathcal{E}$, then $A = B^\perp$,
- the edges of the graph $\langle \Omega_\Gamma, \mathcal{A} \cup \mathcal{E}\rangle$ can be drawn without intersections on a semiplane while the vertices of the graph are ordered according to $<_\Gamma$ on the border of the semiplane,
- the graph $\langle \Omega_\Gamma, \prec_\Gamma \cup \mathcal{A}\rangle$ is acyclic.

**Theorem 8 (1998).** $\mathrm{MCLL} \vdash\ \to \Gamma$ *if and only if there exists a proof net for* $\Gamma$.

# 10 Equivalence

**Definition.** $\mathrm{MCLL} \vdash A \to B$ iff $\mathrm{MCLL} \vdash\ \to A^\perp B$.

**Definition.** $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$ iff $\mathrm{MCLL} \vdash A \to B$ and $\mathrm{MCLL} \vdash B \to A$.

**Lemma 5.**
- $A \underset{\mathrm{MCLL}}{\leftrightarrow} A$.
- *If* $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$, *then* $B \underset{\mathrm{MCLL}}{\leftrightarrow} A$.
- *If* $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$ *and* $B \underset{\mathrm{MCLL}}{\leftrightarrow} C$, *then* $A \underset{\mathrm{MCLL}}{\leftrightarrow} C$.
- *If* $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$ *and* $C \underset{\mathrm{MCLL}}{\leftrightarrow} D$, *then* $A \otimes C \underset{\mathrm{MCLL}}{\leftrightarrow} B \otimes D$.
- *If* $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$ *and* $C \underset{\mathrm{MCLL}}{\leftrightarrow} D$, *then* $A ⅋ C \underset{\mathrm{MCLL}}{\leftrightarrow} B ⅋ D$.
- *If* $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$, *then* $A^\perp \underset{\mathrm{MCLL}}{\leftrightarrow} B^\perp$.

**Definition.** $\sharp\colon \mathrm{Fm} \to \mathbb{Z}$

$$\begin{aligned}
\sharp(p_i) = \sharp(\overline{p_i}) &\rightleftharpoons 0,\\
\sharp(A ⅋ B) &\rightleftharpoons \sharp A + \sharp B + 1,\\
\sharp(A \otimes B) &\rightleftharpoons \sharp A + \sharp B - 1.
\end{aligned}$$

**Lemma 6.** *If* $\mathrm{MCLL} \vdash A \to B$, *then* $\sharp A = \sharp B$.

**Definition.** $\mathrm{at}_0\colon \mathrm{Fm} \to \mathcal{P}(\mathrm{At})$ and $\mathrm{at}_1\colon \mathrm{Fm} \to \mathcal{P}(\mathrm{At})$:

$$\begin{aligned}
&\mathrm{at}_0(C) \rightleftharpoons \{C\} \text{ if } C \in \mathrm{At},\\
&\mathrm{at}_1(C) \rightleftharpoons \{C^\perp\} \text{ if } C \in \mathrm{At},\\
&\mathrm{at}_k(A ⅋ B) = \mathrm{at}_k(A \otimes B) \rightleftharpoons \mathrm{at}_k(A) \cup \mathrm{at}_{(k+1+\sharp A \bmod 2)}(B).
\end{aligned}$$

**Lemma 7.** *If* $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$, *then* $\mathrm{at}_0(A) = \mathrm{at}_0(B)$.

**Theorem 9 (2002).** $A \underset{\mathrm{MCLL}}{\leftrightarrow} p_i$ *if and only if* $\mathrm{at}_0(A) = \{p_i\}$, $\sharp A = 0$, *and* $\sharp C \in \{-1, 0, 1\}$ *whenever* $C$ *is a subformula of* $A$.

**Corollary 2.** *There is a deterministic polynomial time algorithm for the special equivalence problem: given* $A \in \mathrm{Fm}$ *and* $p_i$, *to decide whether* $A \underset{\mathrm{MCLL}}{\leftrightarrow} p_i$.

**Remark.** It is unknown whether the same holds for the problem $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$.