Computing Longest Common Substrings Using Suffix Arrays

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1 Problem Definition
Outline

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2. Suffix Arrays
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2. Suffix Arrays
3. The Algorithm
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3. The Algorithm
4. Conclusions
Part 1
Problem Definition
**Problem** (LCS, Longest Common Substring): Given a collection of $N$ strings $A = \{\alpha_1, \ldots, \alpha_N\}$ and an integer $K$ ($2 \leq K \leq N$) find the longest string $\beta$ that is a substring of at least $K$ strings in $A$. 
**Problem** (LCS, Longest Common Substring): Given a collection of $N$ strings $A = \{\alpha_1, \ldots, \alpha_N\}$ and an integer $K$ ($2 \leq K \leq N$) find the longest string $\beta$ that is a substring of at least $K$ strings in $A$.

- **Tools:** Suffix Arrays
- **Time and Space:** Linear and alphabet-independent
- **Model of Computation:** RAM
Part 2
Suffix Arrays
Useful Definitions

- **Definition (Suffix):** Let \( \omega = \omega_1 \omega_2 \ldots \omega_n \) be an arbitrary string of length \( n \). For each \( i \) \((1 \leq i \leq n)\)

\[
\omega[i..] = \omega_i \omega_{i+1} \ldots \omega_n
\]

is a suffix of \( \omega \).

- **Definition (Lexicographic order):** Suppose we have some order on letters of the alphabet \( \Sigma \). This order can be extended in a standard way to strings over \( \Sigma \): \( \alpha < \beta \) iff either \( \alpha \) is proper prefix of \( \beta \) or \( \alpha[1] = \beta[1], \ldots, \alpha[i] = \beta[i], \alpha[i+1] < \beta[i+1] \).
**Definition** (Suffix Array): Let $\omega$ be an arbitrary string of length $n$. Consider its non-empty suffixes

$$\omega[1..], \omega[2..], \ldots, \omega[n..].$$

and order them lexicographically. Let $SA(i)$ denote the starting position of the suffix appearing on the $i$-th place ($1 \leq i \leq n$):

$$\omega[SA(1)..] < \omega[SA(2)..] < \ldots < \omega[SA(n)..].$$
### An Example of Suffix Array

<table>
<thead>
<tr>
<th>suffixes</th>
<th>SA</th>
<th>sorted suffixes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>mississippi</td>
<td>11</td>
</tr>
<tr>
<td>2</td>
<td>ississippi</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>ssissippi</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>sissippi</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>issippi</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>ssippi</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>sippi</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>ippi</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>ppi</td>
<td>4</td>
</tr>
<tr>
<td>10</td>
<td>pi</td>
<td>6</td>
</tr>
<tr>
<td>11</td>
<td>i</td>
<td>3</td>
</tr>
</tbody>
</table>

**Figure:** String *mississippi*, its suffixes, and the corresponding suffix array.
Why Suffix Arrays?

- A **simple** data structure containing all the necessary information.
Why Suffix Arrays?

- A **simple** data structure containing all the necessary information.
- Many **nice** and **simple** efficient construction algorithms (e.g. Kärkäinen, Sanders [2003]) with **alphabet-independent** time and space complexity.
Part 3
The Algorithm
Our Main Result

Theorem
Let the total length of strings $\alpha_1, \ldots, \alpha_N$ be equal to $L$. Then the answer to the LCS problem can be computed in $O(L)$ time and in $O(L)$ space.
Consider the following example with $N = 3$, $K = 2$:

\[
\begin{align*}
\alpha_1 &= abb \\
\alpha_2 &= cb \\
\alpha_3 &= abc
\end{align*}
\]

Clearly, the answer is $ab$. 
Observation

- The longest common substring for \( K \) strings of our set is the longest common prefix of some suffixes of these strings.
Observation

The longest common substring for $K$ strings of our set is the longest common prefix of some suffixes of these strings.

We calculate the longest common prefix of every $K$ suffixes of different strings and take the longest one; the latter is the answer to the LCS problem.
Combine the strings in $A$ as follows:

$$\alpha = \alpha_1 \$_1 \alpha_2 \$_2 \ldots \alpha_N \$_N.$$  

Here $\$_i$ are special symbols (sentinels) that are different and lexicographically less than other symbols of the initial alphabet $\Sigma$. 
Combine the strings in $A$ as follows:

$$\alpha = \alpha_1$1\alpha_2$2 \ldots \alpha_N$N.$$

Here $\$i$ are special symbols (sentinels) that are different and lexicographically less than other symbols of the initial alphabet $\Sigma$.

**Example:** $\alpha = aab\$1\ cb\$2\ abc\$3$
Definition (Longest Common Prefixes (LCP) array): The array containing lengths of the longest common prefixes for every pair of consecutive suffixes (w.r.t. lexicographical order).

LCP array can be easily constructed in linear time and space.
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- LCP array can be easily constructed in linear time and space.
- We construct the suffix array and the LCP array for $\alpha$. 
Step 2. Example of SA and LCP

String: \texttt{abb}\$\texttt{1 cb}\$\texttt{2 abc}\$\texttt{3}

\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
SA: & 4 & 7 & 11 & 1 & 8 & 3 & 6 & 2 & 9 & 10 & 5 \\
\hline
LCP: & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & \\
\hline
\end{tabular}

suffixes | SA | sorted suffixes | LCP
---|---|---|---
1 | \texttt{abb}\$\texttt{1 cb}\$\texttt{2 abc}\$\texttt{3} | \texttt{$1 cb$2 abc$3} | 0
2 | \texttt{bb}\$\texttt{1 cb}\$\texttt{2 abc}\$\texttt{3} | \texttt{$2 abc$3} | 0
3 | \texttt{b}\$\texttt{1 cb}\$\texttt{2 abc}\$\texttt{3} | \texttt{$3} | 0
4 | \texttt{$1 cb$2 abc$3} | \texttt{abb}\$\texttt{1 cb}\$\texttt{2 abc}\$\texttt{3} | 2
5 | \texttt{cb}\$\texttt{2 abc$3} | \texttt{abc$3} | 0
6 | \texttt{b}\$\texttt{2 abc$3} | \texttt{b}\$\texttt{1 cb}\$\texttt{2 abc$3} | 1
7 | \texttt{$2 abc$3} | \texttt{bb}\$\texttt{1 cb}\$\texttt{2 abc$3} | 1
8 | \texttt{abc$3} | \texttt{bc$3} | 0
9 | \texttt{bc$3} | \texttt{c$3} | 1
10 | \texttt{c$3} | \texttt{cb}\$\texttt{2 abc$3} | 1
11 | \texttt{$3} | \texttt{cb}\$\texttt{2 abc$3} | 

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Further Ideas

The longest prefix of suffixes of $K$ different strings in $A$ is the longest common prefix of suffixes of $K$ different colors in $\alpha$. 

Example:

<table>
<thead>
<tr>
<th>SA</th>
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<th>Suffixes</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>$\text{abb}$</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>$\text{cb}$</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>$\text{abc}$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$\text{abc}$</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>$\text{abc}$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$\text{abc}$</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>$\text{abc}$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$\text{abc}$</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>$\text{abc}$</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>$\text{abc}$</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>$\text{abc}$</td>
</tr>
</tbody>
</table>
Further Ideas

- The longest prefix of suffixes of $K$ different strings in $A$ is the longest common prefix of suffixes of $K$ different colors in $\alpha$.
- Consider $K$ suffixes at positions $i_1, \ldots, i_K$ and assume that $SA[i_1] < SA[i_2] < \ldots < SA[i_K]$. The length of the longest common prefix of these $K$ suffixes is equal to the minimum of $LCP[i_1], \ldots, LCP[i_K - 1]$.
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<th>9</th>
<th>10</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>LCP:</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Suffixes:** $abb_1, cb_2, abc_3$

$abc_3$
Extensions

Theorem

Problem: Given a collection of $N$ strings $A = \{\alpha_1, \ldots, \alpha_N\}$, for each $K$ ($2 \leq K \leq N$) find the longest string $\beta$ that is a substring of at least $K$ strings in $A$. 

Let the total length of strings $\alpha_1, \ldots, \alpha_N$ be equal to $L$. Then the answer to the above problem can be computed in $O(L \cdot \log^* L)$ time and in $O(L)$ space.
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Open Problem

How to compute an inexact longest common substring?
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The End :-)