L-completeness of the Lambek Calculus with the Reversal Operation

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Abstract

We extend the Lambek calculus with rules for a unary operation corresponding to language reversal and prove that this calculus is complete with respect to the class of models on subsets of free semigroups (L-models). We also prove that categorial grammars based on this calculus generate precisely all context-free languages without the empty word.

1 The Lambek Calculus and L-models

We consider the calculus L, introduced in [2]. The set $\Pr = \{p_1, p_2, p_3, \dots\}$ is called the set of *primitive types*. *Types* of L are built from primitive types using three binary connectives: $\backslash$ *(left division)*, $/$ *(right division)*, and $\cdot$ *(multiplication)*; we shall denote the set of all types by Tp. Capital letters $(A, B, \dots)$ range over types. Capital Greek letters (except Σ) range over finite (possibly empty) sequences of types; Λ stands for the empty sequence. Expressions of the form $\Gamma \to C$, where $\Gamma \neq \Lambda$, are called *sequents* of L.

Axioms: $A \to A$.

Rules:

$$\frac{A\Pi \to B}{\Pi \to A \setminus B}\ (\to \setminus),\ \Pi \neq \Lambda \qquad \frac{\Pi \to A \quad \Gamma B \Delta \to C}{\Gamma \Pi (A \setminus B) \Delta \to C}\ (\setminus \to)$$

$$\frac{\Pi A \to B}{\Pi \to B \,/\, A}\ (\to /),\ \Pi \neq \Lambda \qquad \frac{\Pi \to A \quad \Gamma B \Delta \to C}{\Gamma (B \,/\, A) \Pi \Delta \to C}\ (/ \to)$$

$$\frac{\Pi \to A \quad \Delta \to B}{\Pi\Delta \to A \cdot B}\ (\to \cdot) \qquad \frac{\Gamma A B \Delta \to C}{\Gamma (A \cdot B) \Delta \to C}\ (\cdot \to)$$

$$\frac{\Pi \to A \quad \Gamma A \Delta \to C}{\Gamma \Pi \Delta \to C}\ (\text{cut})$$

Published in Proc. LACL 2012, LNCS vol. 7351, 151–160.
The final publication is available at link.springer.com:
http://link.springer.com/chapter/10.1007/978-3-642-31262-5_10

Now let Σ be an alphabet (an arbitrary nonempty set, finite or countable). By Σ^+ we denote the set of all nonempty words over Σ; the set of all words over Σ, including the empty word, is denoted by Σ^*. The set Σ^+ with the operation of word concatenation is the *free semigroup* generated by Σ. Subsets of Σ^+ are called *languages* over Σ. The three connectives of L correspond to three natural operations on languages ($M, N \subseteq \Sigma^+$): $M \cdot N \leftrightharpoons \{uv \mid u \in M, v \in N\}$, $M \setminus N \leftrightharpoons \{u \in \Sigma^+ \mid (\forall v \in M)\, vu \in N\}$, and $N \,/\, M \leftrightharpoons \{u \in \Sigma^+ \mid (\forall v \in M)\, uv \in N\}$ ("$\leftrightharpoons$" here and further means "equals by definition").

An *L-model* is a pair $\mathcal{M} = \langle \Sigma, w \rangle$, where Σ is an alphabet and w is a function that maps Lambek calculus types to languages over Σ, such that $w(A \cdot B) = w(A) \cdot w(B)$, $w(A \setminus B) = w(A) \setminus w(B)$, and $w(B \,/\, A) = w(B) \,/\, w(A)$ for all $A, B \in \mathrm{Tp}$. Obviously, w can be defined on primitive types in an arbitrary way, and then it is uniquely propagated to all types.

A sequent of the form $F \to G$ is considered *true* in a model $\mathcal{M}$ ($\mathcal{M} \vDash F \to G$) if $w(F) \subseteq w(G)$. L-models give sound and complete semantics for L, due to the following theorem:

Theorem 1. *A sequent $F \to G$ is provable in* L *if and only if it is true in all L-models.*

This theorem is proved in [6]; its special case for the product-free fragment (where we keep only types without multiplication) is much easier and appears in [1]. (The notion of truth in an L-model and this theorem can be easily generalized to sequents with more than one type on the left, since $\mathrm{L} \vdash F_1 F_2 \dots F_n \to G$ if and only if $\mathrm{L} \vdash F_1 \cdot F_2 \cdot \dots \cdot F_n \to G$.)

2 The Lambek Calculus with the Reversal Operation (L^{R})

Now let us consider an extra operation on languages, the *reversal.* For $u = a_1 a_2 \dots a_n$ ($a_1, \dots, a_n \in \Sigma$, $n \geq 1$) let $u^{\mathrm{R}} \leftrightharpoons a_n \dots a_2 a_1$, and for $M \subseteq \Sigma^+$ let $M^{\mathrm{R}} \leftrightharpoons \{u^{\mathrm{R}} \mid u \in M\}$. Let us enrich the language of the Lambek calculus with a new unary connective $^{\mathrm{R}}$ (written in the postfix form, A^{R}). We shall denote the extended set of types by Tp^{R}. If $\Gamma = A_1 A_2 \dots A_n$, then $\Gamma^{\mathrm{R}} \leftrightharpoons A_n^{\mathrm{R}} \dots A_2^{\mathrm{R}} A_1^{\mathrm{R}}$.

The notion of L-model is also easily adapted to the new language by adding an additional constraint on w: $w(A^{\mathrm{R}}) = w(A)^{\mathrm{R}}$.

The calculus L^{R} is obtained from L by adding three new rules for $^{\mathrm{R}}$:

$$\frac{\Gamma \to C}{\Gamma^{\mathrm{R}} \to C^{\mathrm{R}}}\ ({}^{\mathrm{R}} \to {}^{\mathrm{R}}) \qquad \frac{\Gamma A^{\mathrm{RR}} \Delta \to C}{\Gamma A \Delta \to C}\ ({}^{\mathrm{RR}} \to)_{\mathrm{E}} \qquad \frac{\Gamma \to C^{\mathrm{RR}}}{\Gamma \to C}\ (\to {}^{\mathrm{RR}})_{\mathrm{E}}$$

It is easy to see that L^{R} is sound with respect to L-models.

Lemma 1. *The calculus* L^{R} *is a conservative extension of* L *(if $F, G \in \mathrm{Tp}$, then $\mathrm{L}^{\mathrm{R}} \vdash F \to G$ if and only if $\mathrm{L} \vdash F \to G$).*

Proof. The "if" part is obvious. The "only if" part follows from L-completeness of L and L-soundness of L^{R}: if $F \to G$ is provable in L^{R}, then it is true in all L-models, and, therefore, is provable in L. □

L-completeness for the product-free fragment of L^{R} is proved in [4] by a modification of Buszkowski's argument [1] (in [4] the reversal connective is called *involution* and denoted by $\breve{}$ instead of $^{\mathrm{R}}$; the calculus is formulated in a different, but equivalent way). In [4] one can also find a proof of L-completeness of the division-free fragment (where only $\cdot$ and $^{\mathrm{R}}$ connectives are kept). We shall prove L-completeness of the whole calculus.

Theorem 2. *A sequent $F \to G$ ($F, G \in \mathrm{Tp}^{\mathrm{R}}$) is provable in L^{R} if and only if it is true in all L-models.*

A variant of this calculus that allows empty antecedents (an extension with the $^{\mathrm{R}}$ connective of L^*, the variant of L without the restriction $\Pi \neq \Lambda$ on the $(\to \backslash)$ and $(\to /)$ rules) is presented in [3]. The calculus L^* itself is complete with respect to L-models allowing empty words in the languages (free monoid models) [7], but L-completeness of its extension with the $^{\mathrm{R}}$ connective is still an open problem.

3 Equivalences in L^{R} and Normal Form for Types

Types A and B are called *equivalent* in L^{R} (denotation: $A \leftrightarrow B$), if $\mathrm{L}^{\mathrm{R}} \vdash A \to B$ and $\mathrm{L}^{\mathrm{R}} \vdash B \to A$. The relation $\leftrightarrow$ is reflexive, symmetric, and transitive (due to the rule (cut)). Using (cut) one can prove that if $\mathrm{L}^{\mathrm{R}} \vdash F_1 \to G_1$, $F_1 \leftrightarrow F_2$, and $G_1 \leftrightarrow G_2$, then $\mathrm{L}^{\mathrm{R}} \vdash F_2 \to G_2$. Also, $\leftrightarrow$ is a *congruence relation,* in the sense of the following lemma (checked explicitly):

Lemma 2. *If $A_1 \leftrightarrow A_2$ and $B_1 \leftrightarrow B_2$, then $A_1 \cdot B_1 \leftrightarrow A_2 \cdot B_2$, $A_1 \backslash B_1 \leftrightarrow A_2 \backslash B_2$, $B_1 / A_1 \leftrightarrow B_2 / A_2$, $A_1^{\mathrm{R}} \leftrightarrow A_2^{\mathrm{R}}$.*

The following lemma is checked explicitly by presenting the corresponding derivations in L^{R}:

Lemma 3. *The following equivalences hold in L^{R}:*

1. $(A \cdot B)^{\mathrm{R}} \leftrightarrow B^{\mathrm{R}} \cdot A^{\mathrm{R}}$;
2. $(A \backslash B)^{\mathrm{R}} \leftrightarrow B^{\mathrm{R}} / A^{\mathrm{R}}$;
3. $(B / A)^{\mathrm{R}} \leftrightarrow A^{\mathrm{R}} \backslash B^{\mathrm{R}}$;
4. $A^{\mathrm{RR}} \leftrightarrow A$.

For $A \in \mathrm{Tp}^{\mathrm{R}}$ we define $tr(A)$ by induction on the number of connectives in A:

1. $tr(p_i) \leftrightharpoons p_i$;

2. $tr(p_i^{\mathrm{R}}) \leftrightharpoons p_i^{\mathrm{R}}$;

3. $tr(A \cdot B) \leftrightharpoons tr(A) \cdot tr(B)$;

4. $tr(A \setminus B) \leftrightharpoons tr(A) \setminus tr(B)$;

5. $tr(B \mathbin{/} A) \leftrightharpoons tr(B) \mathbin{/} tr(A)$;

6. $tr((A \cdot B)^{\mathrm{R}}) \leftrightharpoons tr(B^{\mathrm{R}}) \cdot tr(A^{\mathrm{R}})$;

7. $tr((A \setminus B)^{\mathrm{R}}) \leftrightharpoons tr(B^{\mathrm{R}}) \mathbin{/} tr(A^{\mathrm{R}})$;

8. $tr((B \mathbin{/} A)^{\mathrm{R}}) \leftrightharpoons tr(A^{\mathrm{R}}) \setminus tr(B^{\mathrm{R}})$;

9. $tr(A^{\mathrm{RR}}) \leftrightharpoons tr(A)$.

The following lemma is proved by induction using Lemma 2 and Lemma 3:

Lemma 4. *Any $A \in \mathrm{Tp}^{\mathrm{R}}$ is equivalent to $tr(A)$.*

We call $tr(A)$ the *normal form* of A. In the normal form, the $^{\mathrm{R}}$ connective can appear only on occurrences of primitive types.

4 L-completeness of L^{R} (Proof)

Now we are going to prove Theorem 2 (the "if" part) by contraposition. Let $\mathrm{L}^{\mathrm{R}} \not\vdash F_0 \to G_0$. We need to construct a countermodel for $F_0 \to G_0$, i.e., a model in which this sequent is not true.

Let $\mathrm{Pr}' \leftrightharpoons \mathrm{Pr} \cup \{p^{\mathrm{R}} \mid p \in \mathrm{Pr}\}$, and let L' be the Lambek calculus with Pr' taken as the set of primitive types instead of Pr. Here $^{\mathrm{R}}$ is not a connective, and p^{R} is considered just a new primitive type, independent from p. Obviously, if $\mathrm{L}' \vdash F \to G$, then $\mathrm{L}^{\mathrm{R}} \vdash F \to G$.

Let $F \leftrightharpoons tr(F_0)$, $G \leftrightharpoons tr(G_0)$. Then $\mathrm{L}^{\mathrm{R}} \not\vdash F \to G$, whence $\mathrm{L}' \not\vdash F \to G$. The calculus L' is essentially the same as L, therefore Theorem 1 gives us a structure $\mathcal{M} = \langle \Sigma, w \rangle$ such that $w(F) \not\subseteq w(G)$. The structure $\mathcal{M}$ indeed falsifies $F \to G$, but it is not a model in the sense of our new language: some of the conditions $w(p_i^{\mathrm{R}}) = w(p_i)^{\mathrm{R}}$ might be not satisfied.

Let Φ be the set of all subtypes of $F \to G$ (including F and G themselves; the notion of subtype is understood in the sense of L^{R}). The construction of $\mathcal{M}$ (see [6]) guarantees that $w(A) \neq \varnothing$ for all $A \in \Phi$. This is the only specific property of $\mathcal{M}$ we shall need.

We introduce an inductively defined counter $f(A)$, $A \in \Phi$: $f(p_i) \leftrightharpoons 1$, $f(p_i^{\mathrm{R}}) \leftrightharpoons 1$, $f(A \cdot B) \leftrightharpoons f(A) + f(B) + 10$, $f(A \setminus B) \leftrightharpoons f(B)$, $f(B \mathbin{/} A) \leftrightharpoons f(B)$. Let $K \leftrightharpoons \max\{f(A) \mid A \in \Phi\}$, $N \leftrightharpoons 2K + 25$ (N should be odd, greater then K, and big enough itself).

Let $\Sigma_1 \leftrightharpoons \Sigma \times \{1, \ldots, N\}$. We shall denote the pair $\langle a, j \rangle \in \Sigma_1$ by $a^{(j)}$. Elements of Σ and Σ_1 will be called *letters* and *symbols* respectively. A symbol can be *even* or *odd* depending on the parity of the superscript. Consider a homomorphism $h \colon \Sigma^+ \to \Sigma_1^+$, defined as follows: $h(a) \leftrightharpoons a^{(1)} a^{(2)} \ldots a^{(N)}$ ($a \in \Sigma$),

$h(a_1 \dots a_n) \leftrightharpoons h(a_1) \dots h(a_n)$. Let $P \leftrightharpoons h(\Sigma^+) = \{a_1^{(1)} \dots a_1^{(N)} \dots a_n^{(1)} \dots a_n^{(N)} \mid n \geq 1,\ a_i \in \Sigma\}$. Note that h is a bijection between Σ^+ and P.

Lemma 5. *For all $M, N \subseteq \Sigma^+$ we have*

1. $h(M \cdot N) = h(M) \cdot h(N)$;
2. *if $M \neq \varnothing$, then $h(M \setminus N) = h(M) \setminus h(N)$ and $h(N \,/\, M) = h(N) \,/\, h(M)$.*

Proof.

1. By the definition of a homomorphism.
2. $\boxed{\subseteq}$ Let $u \in h(M \setminus N)$. Then $u = h(u')$ for some $u' \in M \setminus N$. For all $v' \in M$ we have $v'u' \in N$. Take an arbitrary $v \in h(M)$, $v = h(v')$ for some $v' \in M$. Since $u' \in M \setminus N$, $v'u' \in N$, whence $vu = h(v')h(u') = h(v'u') \in h(N)$. Therefore $u \in h(M) \setminus h(N)$.

   $\boxed{\supseteq}$ Let $u \in h(M) \setminus h(N)$. First we claim that $u \in P$. Suppose the contrary: $u \notin P$. Take $v' \in M$ (M is nonempty by assumption). Since $v = h(v') \in P$, $vu \notin P$. On the other hand, $vu \in h(N) \subseteq P$. Contradiction. Now, since $u \in P$, $u = h(u')$ for some $u' \in \Sigma^+$. For an arbitrary $v' \in M$ and $v \leftrightharpoons h(v')$ we have $h(v'u') = vu \in h(N)$, whence $v'u' \in N$, whence $u' \in M \setminus N$. Therefore, $u = h(u') \in h(M \setminus N)$.

   The / case is handled symmetrically.

$\square$

We construct a new model $\mathcal{M}_1 = \langle \Sigma_1, w_1 \rangle$, where $w_1(z) \leftrightharpoons h(w(z))$ ($z \in \mathrm{Pr}'$). Due to Lemma 5, $w_1(A) = h(w_1(A))$ for all $A \in \Phi$, whence $w_1(F) = h(w(F)) \not\subseteq h(w(G)) = w_1(G)$ ($\mathcal{M}_1$ is also a countermodel in the language without $^{\mathrm{R}}$).

Now we introduce several auxiliary subsets of Σ_1^+ (by $\mathrm{Subw}(M)$ we denote the set of all nonempty subwords of words from M, i.e. $\mathrm{Subw}(M) \leftrightharpoons \{u \in \Sigma_1^+ \mid (\exists v_1, v_2 \in \Sigma_1^*)\, v_1 u v_2 \in M\}$):
$T_1 \leftrightharpoons \{u \in \Sigma_1^+ \mid u \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})\}$;
$T_2 \leftrightharpoons \{u \in \mathrm{Subw}(P \cup P^{\mathrm{R}}) \mid \text{the first or the last symbol of } u \text{ is even}\}$;
$E \leftrightharpoons \{u \in \mathrm{Subw}(P \cup P^{\mathrm{R}}) - (P \cup P^{\mathrm{R}}) \mid \text{both the first symbol and the last symbol of } u \text{ are odd}\}$.

The sets P, P^{R}, T_1, T_2, and E form a partition of Σ_1^+ into nonintersecting parts. For example, $a^{(1)} b^{(10)} a^{(2)} \in T_1$, $a^{(N)} b^{(1)} \dots b^{(N-1)} \in T_2$, $a^{(7)} a^{(6)} a^{(5)} \in E$ ($a, b \in \Sigma$).

Let $T \leftrightharpoons T_1 \cup T_2$, $T_i(k) \leftrightharpoons \{u \in T_i \mid |u| \geq k\}$ ($i = 1, 2$, $|u|$ is the length of u), $T(k) \leftrightharpoons T_1(k) \cup T_2(k) = \{u \in T \mid |u| \geq k\}$.

Note that if the first or the last symbol of u is even, then it belongs to T, no matter whether it belongs to $\mathrm{Subw}(P \cup P^{\mathrm{R}})$.

The index k (possibly with subscripts) here and further ranges from 1 to K. For all k we have $T(k) \supseteq T(K)$.

Lemma 6.

1. $P \cdot P \subseteq P$, $P^{\mathrm{R}} \cdot P^{\mathrm{R}} \subseteq P^{\mathrm{R}}$;
2. $T^{\mathrm{R}} = T$, $T(k)^{\mathrm{R}} = T(k)$;
3. $P \cdot P^{\mathrm{R}} \subseteq T(K)$, $P^{\mathrm{R}} \cdot P \subseteq T(K)$;
4. $P \cdot T \subseteq T(K)$, $T \cdot P \subseteq T(K)$;
5. $P^{\mathrm{R}} \cdot T \subseteq T(K)$, $T \cdot P^{\mathrm{R}} \subseteq T(K)$;
6. $T \cdot T \subseteq T$;

Proof.

1. Obvious.
2. Directly follows from our definitions.
3. Any element of $P \cdot P^{\mathrm{R}}$ or $P^{\mathrm{R}} \cdot P$ does not belong to $\mathrm{Subw}(P \cup P^{\mathrm{R}})$ and its length is at least $2N > K$. Therefore it belongs to $T_1(K) \subseteq T(K)$.
4. Let $u \in P$ and $v \in T$. If $v \in T_1$, then uv is also in T_1. Let $v \in T_2$. If the last symbol of v is even, then $uv \in T$. If the last symbol of v is odd, then $uv \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})$, whence $uv \in T_1 \subseteq T$. Since $|uv| > |u| \geq N > K$, $uv \in T(K)$.

   The claim $T \cdot P \subseteq T$ is handled symmetrically.
5. $P^{\mathrm{R}} \cdot T = P^{\mathrm{R}} \cdot T^{\mathrm{R}} = (T \cdot P)^{\mathrm{R}} \subseteq T(K)^{\mathrm{R}} = T(K)$. $T \cdot P^{\mathrm{R}} = T^{\mathrm{R}} \cdot P^{\mathrm{R}} = (P \cdot T)^{\mathrm{R}} \subseteq T(K)^{\mathrm{R}} = T(K)$.
6. Let $u, v \in T$. If at least one of these two words belongs to T_1, then $uv \in T_1$. Let $u, v \in T_2$. If the first symbol of u or the last symbol of v is even, then $uv \in T$. In the other case u ends with an even symbol, and v starts with an even symbol. But then we have two consecutive even symbols in uv, therefore $uv \in T_1$.

□

Let us call words of the form $a^{(i)}a^{(i+1)}a^{(i+2)}$, $a^{(N-1)}a^{(N)}b^{(1)}$, and $a^{(N)}b^{(1)}b^{(2)}$ ($a, b \in \Sigma$, $1 \leq i \leq N-2$) *valid triples of type I* and their reversals (namely, $a^{(i+2)}a^{(i+1)}a^{(i)}$, $b^{(1)}a^{(N)}a^{(N-1)}$, and $b^{(2)}b^{(1)}a^{(N)}$) *valid triples of type II.* Note that valid triples of type I (resp., of type II) are the only possible three-symbol subwords of words from P (resp., P^{R}).

Lemma 7. *A word u of length at least three is a subword of a word from $P \cup P^{\mathrm{R}}$ if and only if any three-symbol subword of u is a valid triple of type I or II.*

Proof. The nontrivial part is "if". We proceed by induction on $|u|$. Induction base ($|u| = 3$) is trivial. Let u be a word of length $m+1$ satisfying the condition and let $u = u'x$ ($x \in \Sigma_1$). By induction hypothesis ($|u'| = m$), $u' \in \mathrm{Subw}(P \cup P^{\mathrm{R}})$. Let $u' \in \mathrm{Subw}(P)$ (the other case is handled symmetrically); u' is a subword of some word $v \in P$. Consider the last three symbols of u. Since the first two of them also belong to u', this three-symbol word is a valid triple of type I, not type II. If it is of the form $a^{(i)}a^{(i+1)}a^{(i+2)}$ or $a^{(N)}b^{(1)}b^{(2)}$, then x coincides with the symbol next to the occurrence of u' in v, and therefore $u = u'x$ is also a subword of v. If it is of the form $a^{(N-1)}a^{(N)}b^{(1)}$, then, provided $v = v_1u'v_2$, v_1u' is also an element of P, and so is the word $v_1u'b^{(1)}b^{(2)}\dots b^{(N)}$, which contains $u = u'b^{(1)}$ as a subword. Thus, in all cases $u \in \mathrm{Subw}(P)$. □

Now we construct one more model $\mathcal{M}_2 = \langle \Sigma_1, w_2 \rangle$, where $w_2(p_i) \leftrightharpoons w_1(p_i) \cup w_1(p_i^{\mathrm{R}})^{\mathrm{R}} \cup T$, $w_2(p_i^{\mathrm{R}}) \leftrightharpoons w_1(p_i)^{\mathrm{R}} \cup w_1(p_i^{\mathrm{R}}) \cup T$. This model is a model even in the sense of the enriched language. To finish the proof, we need to check that $\mathcal{M}_2 \not\models F \to G$.

Lemma 8. *For any $A \in \Phi$ the following holds:*

1. $w_2(A) \subseteq P \cup P^{\mathrm{R}} \cup T$;
2. $w_2(A) \supseteq T(f(A))$;
3. $w_2(A) \cap P = w_1(A)$ *(in particular, $w_2(A) \cap P \neq \varnothing$);*
4. $w_2(A) \cap P^{\mathrm{R}} = w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}}$ *(in particular, $w_2(A) \cap P^{\mathrm{R}} \neq \varnothing$).*

Proof. We prove all the statements simultaneously by induction on type A. The induction base is trivial. Further we shall refer to the i-th statement of the induction hypothesis ($i = 1, 2, 3, 4$) as "IH-i".

1. Consider three possible cases.

a) $A = B \cdot C$. Then $w_2(A) = w_2(B) \cdot w_2(C) \subseteq (P \cup P^{\mathrm{R}} \cup T) \cdot (P \cup P^{\mathrm{R}} \cup T) \subseteq P \cup P^{\mathrm{R}} \cup T$ (Lemma 6).

b) $A = B \setminus C$. Suppose the contrary: in $w_2(A)$ there exists an element $u \in E$. Then $vu \in w_2(C)$ for any $v \in w_2(B)$. We consider several subcases and show that each of those leads to a contradiction.

i) $u \in \mathrm{Subw}(P)$, and the superscript of the first symbol of u is not 1. Let the first symbol of u be $a^{(i)}$. Note that i is odd and $i > 2$. Take $v = a^{(3)} \dots a^{(N)}a^{(1)} \dots a^{(i-1)}$. The word v has length at least $N \geq K$ and ends with an even symbol, therefore $v \in T(K) \subseteq T(f(B)) \subseteq w_2(B)$ (IH-2). On the other hand, $vu \in \mathrm{Subw}(P)$ and the first symbol and the last symbol of vu are odd. Therefore, $vu \in E$ and $vu \in w_2(C)$, but $w_2(C) \cap E = \varnothing$ (IH-1). Contradiction.

ii) $u \in \mathrm{Subw}(P)$, and the first symbol of u is $a^{(1)}$ (then the superscript of the last symbol of u is not N, because otherwise $u \in P$). Take $v \in w_2(B) \cap P$ (this set is nonempty due to IH-3). The first and the last symbol of vu is odd, and $vu \in \mathrm{Subw}(P) - P$, therefore $vu \in E$. Contradiction.

iii) $u \in \mathrm{Subw}(P^{\mathrm{R}})$, and the superscript of the first symbol of u is not N (the first symbol of u is $a^{(i)}$, i is odd). Take $v = a^{(N-2)} \dots a^{(1)}a^{(N)} \dots a^{(i+1)} \in T(K) \subseteq w_2(B)$. Again, $vu \in E$.

iv) $u \in \mathrm{Subw}(P^{\mathrm{R}})$, and the first symbol of u is $a^{(N)}$. Take $v \in w_2(B) \cap P^{\mathrm{R}}$ (nonempty due to IH-4). $vu \in E$.

c) $A = C \,/\, B$. Proceed symmetrically.

2. Consider three possible cases.

a) $A = B \cdot C$. Let $k_1 \leftrightharpoons f(B)$, $k_2 \leftrightharpoons f(C)$, $k \leftrightharpoons k_1 + k_2 + 10 = f(A)$. Due to IH-2, $w_2(B) \supseteq T(k_1)$ and $w_2(C) \supseteq T(k_2)$. Take $u \in T(k)$. We have to prove that $u \in w_2(A)$. Consider several subcases.

i) $u \in T_1(k)$. By Lemma 7 ($|u| \geq k > 3$ and $u \notin \mathrm{Subw}(P \cup P^{\mathrm{R}})$) in u there is a three-symbol subword xyz that is not a valid triple of type I or II. Divide the word u into two parts, $u = u_1 u_2$, such that $|u_1| \geq k_1 + 5$, $|u_2| \geq k_2 + 5$. If needed, shift the border between parts by one symbol to the left or to the right, so that the subword xyz lies entirely in one part. Let this part be u_2 (the other case is handled symmetrically). Then $u_2 \in T_1(k_2)$. If u_1 is also in T_1, then the proof is finished. Consider the other case. Note that in any word from $\mathrm{Subw}(P \cup P^{\mathrm{R}})$ among any three consecutive symbols at least one is even. Shift the border to the left by at most 2 symbols to make the last symbol of u_1 even. Then $u_1 \in T(k_1)$, and u_2 remains in $T_1(k_2)$. Thus $u = u_1 u_2 \in T(k_1) \cdot T(k_2) \subseteq w_2(B) \cdot w_2(C) = w_2(A)$.

ii) $u \in T_2(k)$. Let u end with an even symbol (the other case is symmetric). Divide the word u into two parts, $u = u_1 u_2$, $|u_1| \geq k_1 + 5$, $u_2 \geq k_2 + 5$, and shift the border (if needed), so that the last symbol of u_1 is even. Then both u_1 and u_2 end with an even symbol, and therefore $u_1 \in T(k_1)$ and $u_2 \in T(k_2)$.

b) $A = B \setminus C$. Let $k \leftrightharpoons f(C) = f(A)$. By IH-2, $w_2(C) \supseteq T(k)$. Take $u \in T(k)$ and an arbitrary $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup T$. By Lemma 6, statements 4–6, $vu \in (P \cup P^{\mathrm{R}} \cup T) \cdot T \subseteq T$, and since $|vu| > |u| \geq k$, $vu \in T(k) \subseteq w_2(C)$. Thus $u \in w_2(A)$.

c) $A = C \,/\, B$. Symmetrically.

3. Consider three possible cases.

a) $A = B \cdot C$.

$\boxed{\supseteq}$ $u \in w_1(A) = w_1(B) \cdot w_1(C) \subseteq w_2(B) \cdot w_2(C) = w_2(A)$ (IH-3); $u \in P$.

$\boxed{\subseteq}$ Suppose $u \in P$ and $u \in w_2(A) = w_2(B) \cdot w_2(C)$. Then $u = u_1 u_2$, where $u_1 \in w_2(B)$ and $u_2 \in w_2(C)$. First we claim that $u_1 \in P$. Suppose the contrary, $u_1 \notin P$. By IH-1, $u_1 \in P^{\mathrm{R}} \cup T$, $u_2 \in P \cup P^{\mathrm{R}} \cup T$, and therefore $u = u_1 u_2 \in (P^{\mathrm{R}} \cup T) \cdot (P \cup P^{\mathrm{R}} \cup T) \subseteq P^{\mathrm{R}} \cup T$ (Lemma 6, statements 1, 3–6). Hence $u \notin P$. Contradiction. Thus, $u_1 \in P$. Similarly, $u_2 \in P$, and by IH-3 we obtain $u_1 \in w_1(B)$ and $u_2 \in w_1(C)$, whence $u = u_1 u_2 \in w_1(A)$.

b) $A = B \setminus C$.

$\boxed{\supseteq}$ Take $u \in w_1(B \setminus C)$. For any $v \in w_1(B)$ we have $vu \in w_1(C)$. We claim that $u \in w_2(B \setminus C)$. Take $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup T$ (IH-1). If $v \in P$, then $v \in w_1(B)$ (IH-3), and $vu \in w_1(C) \subseteq w_2(C)$ (IH-3). If $v \in P^{\mathrm{R}} \cup T$, then $vu \in (P^{\mathrm{R}} \cup T) \cdot P \subseteq T(K) \subseteq w_2(C)$ (Lemma 6, statements 3 and 4, and IH-2). Therefore, $u \in w_2(B) \setminus w_2(C) = w_2(B \setminus C)$; also we have $u \in P$, since $w_1(B \setminus C) \subseteq P$.

$\boxed{\subseteq}$ If $u \in w_2(B \setminus C)$ and $u \in P$, then for any $v \in w_1(B) \subseteq w_2(B)$ we have $vu \in w_2(C)$. Since $v, u \in P$, $vu \in P$. By IH-3, $vu \in w_1(C)$. Thus $u \in w_1(B \setminus C)$.

c) $A = C \,/\, B$. Symmetrically.

4. Consider three cases.

a) $A = B \cdot C$. Then $tr(A^{\mathrm{R}}) = tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}})$.

$\boxed{\supseteq}$ $u \in w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}}))^{\mathrm{R}} = \big(w_1(tr(C^R)) \cdot w_1(tr(B^{\mathrm{R}}))\big)^{\mathrm{R}} = w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \cdot w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(B) \cdot w_2(C) = w_2(A)$ (IH-4); $u \in P^{\mathrm{R}}$.

$\boxed{\subseteq}$ Let $u \in P^{\mathrm{R}}$ and $u \in w_2(A) = w_2(B) \cdot w_2(C)$. Then $u = u_1 u_2$, where $u_1 \in w_2(B)$, $u_2 \in w_2(C)$. We claim that $u_1, u_2 \in P^{\mathrm{R}}$. Suppose the contrary: $u_1 \notin P^{\mathrm{R}}$. Then $u_1 \in P \cup T$ (IH-1), $u_2 \in P \cup P^{\mathrm{R}} \cup T$, whence $u = u_1 u_2 \in (P \cup T) \cdot (P \cup P^{\mathrm{R}} \cup T) \subseteq P \cup T$. Contradiction ($u \in P^{\mathrm{R}}$). Thus, $u_1 \in P^{\mathrm{R}}$, and therefore $u_2 \in P^R$, and, using IH-4, we obtain $u_1 \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$, $u_2 \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$. Hence $u = u_1 u_2 \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \cdot w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} = \big(w_1(tr(C^{\mathrm{R}})) \cdot w_1(tr(B^{\mathrm{R}}))\big)^{\mathrm{R}} = w_1(tr(C^{\mathrm{R}}) \cdot tr(B^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(A^{\mathrm{R}}))^{\mathrm{R}}$.

b) $A = B \setminus C$. Then $tr(A^{\mathrm{R}}) = tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}})$.

$\boxed{\supseteq}$ Let $u \in w_1(tr(C^{\mathrm{R}}) \,/\, tr(B^{\mathrm{R}}))^{\mathrm{R}} = w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \setminus w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$, so for every $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$ we have $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$. We claim that $u \in w_2(B \setminus C)$. Take an arbitrary $v \in w_2(B) \subseteq P \cup P^{\mathrm{R}} \cup T$ (IH-1). If $v \in P^{\mathrm{R}}$, then $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}}$ (IH-4), whence $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(C)$.
If $v \in P \cup T$, then (since $u \in P^{\mathrm{R}}$) we have $vu \in (P \cup T) \cdot P^{\mathrm{R}} \subseteq T(K) \subseteq w_2(C)$ (Lemma 6 and IH-2).

$\boxed{\subseteq}$ If $u \in w_2(B \setminus C)$ and $u \in P^{\mathrm{R}}$, then for any $v \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \subseteq w_2(B)$ we have $vu \in w_2(C)$. Since $v, u \in P^R$, $vu \in P^{\mathrm{R}}$, therefore $vu \in w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}}$ (IH-4). Thus $u \in w_1(tr(B^{\mathrm{R}}))^{\mathrm{R}} \setminus w_1(tr(C^{\mathrm{R}}))^{\mathrm{R}} = w_1(A^{\mathrm{R}})^{\mathrm{R}}$.

c) $A = C \,/\, B$. Symmetrically.

This completes the proof of Lemma 8. □

Since $w_1(F) \not\subseteq w_1(G)$, there exists an element u_0 such that $u_0 \in w_1(F)$ and $u_0 \notin w_1(G)$. Since $u_0 \in P$, $u_0 \in w_2(F)$ and $u_0 \notin w_2(G)$. Therefore, $w_2(F) \not\subseteq w_2(G)$. Since $F_0 \leftrightarrow F$, $G_0 \leftrightarrow G$, and L^{R} is L-sound, we see that $w_2(F_0) = w_2(F)$, $w_2(G_0) = w_2(G)$, and $\mathcal{M}_2$ is a countermodel for $F_0 \to G_0$. This completes the proof of Theorem 2.

Note that we have constructed a countermodel (in the sense of the extended language) for any sequent $F \to G$ that is not provable in L' (this could be potentially weaker than $\mathrm{L}^{\mathrm{R}} \nvdash F \to G$). Thus we get the following statement:

Lemma 9. $\mathrm{L}^{\mathrm{R}} \vdash A_1 \dots A_n \to B$ *if and only if* $\mathrm{L}' \vdash tr(A_1) \dots tr(A_n) \to tr(B)$.

5 L^{R}-grammars

The Lambek calculus and its variants are used for describing formal languages via Lambek categorial grammars. An L-*grammar* is a triple $\mathcal{G} = \langle \Sigma, H, \rhd \rangle$, where Σ is a finite alphabet, $H \in \mathrm{Tp}$, and $\rhd$ is a finite correspondence between Tp and Σ ($\rhd \subset \mathrm{Tp} \times \Sigma$). The *language generated by* $\mathcal{G}$ is the set of all nonempty words $a_1 \dots a_n$ over Σ for which there exist types $B_1, \dots, B_n$ such that $\mathrm{L} \vdash B_1 \dots B_n \to H$ and $B_i \triangleright a_i$ for all $i \le n$. We denote this language by $\mathfrak{L}(\mathcal{G})$. This class of grammars is *weakly equivalent* to the class of context-free grammars (without ϵ-rules) in the following sense:

Theorem 3. *A formal language without the empty word is context-free if and only if it is generated by some* L*-grammar.* [5]

By modifying our definition in a natural way one can introduce the notion of L^{R}-grammar. These grammars also generate precisely all context-free languages without the empty word:

Theorem 4. *A formal language without the empty word is context-free if and only if it is generated by some* L^{R}*-grammar.*

Proof. The "only if" part follows directly from Theorem 3 due to conservativity of L^{R} over L (Lemma 1).

Let us prove the "if" part. Let $M = \mathfrak{L}(\mathcal{G})$ for some L^{R}-grammar $\mathcal{G}$. Let $\mathcal{G}'$ be the grammar obtained from $\mathcal{G}$ by replacing all the types with their normal forms and considering it as an L'-grammar. By Lemma 9, $\mathfrak{L}(\mathcal{G}') = \mathfrak{L}(\mathcal{G}) = M$, and this language is context-free due to Theorem 3 (because L' and L are essentially the same calculus). □

Acknowledgments

I am grateful to Prof. Mati Pentus for fruitful discussions and constant attention to my work. I am also grateful to Prof. Sergei Adian for inspiring techniques of working with words over an alphabet given in his lectures and papers.

This research was supported by the Russian Foundation for Basic Research (grant 11-01-00281-a), by the Presidential Council for Support of Leading Research Schools (grant NSh-65648.2010.1) and by the Scientific and Technological Cooperation Programme Switzerland–Russia (STCP-CH-RU, project "Computational Proof Theory").

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