

Short lists with short programs in short time

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Abstract—Given a machine U , a c -short program for x is a string p such that $U(p) = x$ and the length of p is bounded by $c +$ (the length of a shortest program for x). We show that for any universal machine, it is possible to compute in polynomial time on input x a list of polynomial size guaranteed to contain a $O(\log |x|)$ -short program for x . We also show that there exist computable functions that map every x to a list of size $O(|x|^2)$ containing a $O(1)$ -short program for x and this is essentially optimal because we prove that such a list must have size $\Omega(|x|^2)$. Finally we show that for some machines, computable lists containing a shortest program must have length $\Omega(2^{|x|})$.

Keywords: list-approximator, Kolmogorov complexity, on-line matching, expander graph

I. INTRODUCTION

The Kolmogorov complexity of a string x is the length of a shortest program computing it. Determining the Kolmogorov complexity of a string is a canonical example of a function that is not computable. Closely related, and non-computable as well, is the problem of actually producing a shortest program for x . It is natural to ask if tasks marred by such an impossibility barrier can be effectively solved at least in some approximate sense. This issue has been investigated for Kolmogorov complexity in various ways. First of all, it is well-known that the Kolmogorov complexity can be effectively approximated from above. A different type of approximation is given by what is typically called *list computability* in algorithms and complexity theory and *traceability (enumeration)* in computability theory. For this type of approximation, one would like to compute a list of “suspects” for the result of the function with the guarantee that the actual result is in the list. Of course, the shorter the list is, the better is the approximation.

The list approximability of the Kolmogorov complexity, $C(x)$, has been studied by Beigel et al. [BBF⁺06]. They observe that $C(x)$ can be approximated by a list of size $(n - a)$ for every constant a , where $n = |x|$. On the other hand, they show that, for every universal machine U , there is a constant c such that for infinitely many strings x (in fact for at least one x at each sufficiently large length n), any

computable list containing $C_U(x)$ must have size larger than n/c .

In this paper we study list approximability for the problem of producing short programs. In order to describe our results, we need several formal definitions.

A machine U is *optimal* if $C_U(x|y) \leq C_V(x|y) + O(1)$ for all machines V (where the constant $O(1)$ may depend on V). An optimal machine U is *standard*¹, if for every machine V there is a total computable function t such that for all p, y : $U(t(p), y) = V(p, y)$ and $|t(p)| = |p| + O(1)$. For results that hold in polynomial time, we additionally assume these functions t run in time polynomial in $|p|$. Let $U(p)$ stand for $U(p, \text{the empty string})$ and $C_U(x)$ for $C_U(x|\text{the empty string})$. A c -short program for x with respect to U is a string p that satisfies $U(p) = x$ and $|p| \leq C_U(x) + c$.

Given an optimal machine U , a *list-approximator* for c -short programs is a function f that on every input x outputs a finite list of strings such that at least one of the elements in the list is a c -short program for x on U . Let $|f(x)|$ denote the number of elements in the list $f(x)$. Obviously, for every optimal U , there is a (trivial) computable list-approximator f such that $|f(x)| \leq 2^{|x|+O(1)}$.

The question we study is how small can $|f(x)|$ be for computable list-approximators f for c -short programs, where c is a constant or $O(\log |x|)$. At first glance it seems that in both cases $|f(x)|$ must be exponential in $|x|$. Surprisingly, this is not the case. We prove that there is a computable approximator with list of size $O(|x|^2)$ for c -short programs for some constant c depending on the choice of the standard machine U . And we show that this bound is tight. We show also that there is a *polynomial time computable* approximator with list of size $\text{poly}(|x|)$ for c -short programs for $c = O(\log |x|)$.

We start with the positive results, i.e., the upper bounds. We show that for every standard machine, there exists a list-approximator for $O(1)$ -short programs, with lists of *quadratic* size.

Theorem 1.1. *For every standard machine U there exists a*

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¹This notion was introduced by Schnorr [Sch75], and he called such machines *optimal Gödel numberings* (of the family of all computable functions from strings to strings). We use a different term to distinguish between optimal functions in Kolmogorov’s sense and Schnorr’s sense

computable function f that for any x produces a list with $O(|x|^2)$ many elements containing a program p for x of length $|p| = C_U(x) + O(1)$.

If we allow $O(\log|x|)$ -short programs we can construct lists of polynomial size in *polynomial time*.

Theorem I.2. *For every standard machine U , there exists a polynomial-time computable function f that for any x produces a list with $\text{poly}(|x|)$ many elements containing a program for x of length $C_U(x) + O(\log|x|)$.²*

Now we move to the lower bounds. We show that the list size in Theorem I.1 is optimal: it is not possible to compute lists of subquadratic size that contain a $O(1)$ -short program.

Theorem I.3. *For all $c > 0$, for every optimal U , for every computable f that is a list-approximator for c -short programs,*

$$|f(x)| \geq \Omega(|x|^2/c^2),$$

for infinitely many x . (The constant hidden in Ω -notation depends on the function f and machine U .)

The next lower bound shows that, at least for some standard machine U , we can not compute lists of subexponential size containing a program of length exactly $C(x)$.

Theorem I.4. *There exists a standard³ machine U such that for every computable f that is a list-approximator for O -short programs,*

$$|f(x)| \geq 2^{|x| - O(1)},$$

for infinitely many x . (The constant hidden in O -notation depends on the function f .)⁴

The construction of U can be easily adapted to show that for every c there is a standard machine U_c satisfying the theorem for c -short programs. This exponential lower bound does not hold for all standard machines (see Remark 1 below).

Technical overview. A c -short program of a string x is a compressed representation of x ; it is also a string of (close to) minimal length that retains all the randomness in x . The idea that comes to mind to approach list-approximability of c -short programs is to use randomness extractors. Kolmogorov complexity extraction has been studied before [FHP⁺11], [HPV11], [Zim10b], [Zim11b], [Zim11a] (see also the survey paper [Zim10a]), and indeed randomness extractors for a constant number of independent sources have been employed for this task. For the list-approximability of c -short programs, it seems natural to use seeded extractors,

²Recently, Jason Teutsch [Teu12] has improved this result replacing $O(\log|x|)$ by $O(1)$.

³The construction implies the existence of such U with a stronger universality property: for every machine V there exists a string w_V such that $U(w_V p) = V(p)$ for all p .

⁴Independently, this result was obtained by Frank Stephan (personal communication).

because by iterating over all possible seeds, one obtains a list containing the optimally compressed string. The problem is that we need an extractor with logarithmic seed (because we want a list of polynomial size) and zero entropy loss (because we want the compressed string to be a program for x , i.e., to contain enough information so that x can be reconstructed from it) and such extractors have not yet been shown to exist. Perhaps surprisingly, simpler graphs satisfying less demanding combinatorial constraints than extractors graphs, are good enough for the list approximation of c -short programs (and also for this type of “list”-extraction of Kolmogorov complexity). Inspired by [MRS11], we use graphs that allow on-line matching. These are unbalanced bipartite graphs, which, in their simplest form, have $\text{LEFT} = \{0, 1\}^n$, $\text{RIGHT} = \{0, 1\}^{k+\text{small overhead}}$, and left degree = $\text{poly}(n)$, and which permit on-line matching up to size $K = 2^k$. This means that any set of K left nodes, each one requesting to be matched to some adjacent right node that was not allocated earlier, can be satisfied in the on-line manner (i.e., the requests arrive one by one and each request is satisfied before seeing the next one; in some of our proofs we will allow a small number of requests to be discarded, but this should also happen before the next request arrives). The correspondence to our problem is roughly that left strings are the strings that we want to compress, and for any left x we seek its compressed form among its right neighbors. To understand this correspondence, let us consider the easier situation of producing a short list containing a c -short program for x in case the program knows $n = |x|$. Also, let us assume that the Kolmogorov complexity of x is k . We start an enumeration of strings produced by k -bit programs, and when a string of length n is enumerated (eventually, x), we use the on-line matching process and find it a match among its right neighbors, i.e., we compress it to length $k + \text{small overhead}$. To decompress, we start with a right node (the compressed string) and we re-play the enumeration and the matching process and see which left node has been matched to it; we output this left node. The compressed program for x is among its right neighbors and, therefore the set of right neighbors of x is the desired list of polynomial size. Now, in fact, neither n nor k are known, and therefore we actually need the bipartite graph to be infinite and the matching requests for a left node x to include the desired ($k + \text{overhead}$) length for the matching right node. With this modification, it turns out that the construction of a list approximator f for c -short programs is equivalent to the construction of an infinite bipartite graph G that can satisfy the on-line matching requests with overhead equal to $c + O(1)$. The size of the list $f(x)$ is equal to the degree of x in G . Such an infinite graph is obtained by taking the union of finite graphs of the type described above. In order for a finite graph to allow matching, it needs to have good expansion properties. It turns out that it is enough if left

subsets of size $K/O(1)$ expand to size K . To obtain the graph required in Theorem I.1, we use the probabilistic method (actually to get quadratic left degree, we need to refine the construction sketched above). The explicit graph required in Theorem I.2 is obtained from the disperser in [TSUZ07].

The lower bound in Theorem I.3 is established via the equivalence between list approximability and graphs with on-line matching mentioned above. It is next observed that bipartite graphs capable of satisfying even off-line matching need to have a certain expansion property and this imposes a lower bound on the left degree, which, as we have seen, corresponds to the list size.

The exponential lower bound in Theorem I.4 is shown using a connection with a type of Kolmogorov complexity that is less known, *total conditional Kolmogorov complexity*, and then building strings with large such complexity using a game-theoretic approach. (The game-based technique in recursion theory was introduced by Lachlan [Lac70] and further developed by A.Muchnik and others [MMSV10], [Ver08], [MSV12].)

Paper organization The connection between list-approximability and graphs with on-line matching is studied in Section II. The upper bounds, i.e., Theorem I.1 and Theorem I.2, are proved in Section III. The lower bounds, i.e., Theorem I.3 and Theorem I.4, are proved in Section IV. In Section V, we observe that our technique can be used to improve Muchnik’s Theorem [Muc02] (see also [MRS11], [Mus11]), and a result concerning distinguishing complexity [BFL01].

II. LIST APPROXIMATORS FOR SHORT PROGRAMS AND ON-LINE MATCHING

We show that the problem of constructing approximators for short programs is equivalent to constructing families of bipartite graphs which permit on-line matching, a notion introduced in a somewhat different form in [MRS11].

Let a bipartite graph $G = (L, R, E \subseteq L \times R)$ be given, where the set L of left nodes and the set R of right nodes consist of binary strings. Assume that we receive “requests for matching” in the graph, each request having the form (a binary string $x \in L$, a natural number k). Such a request is satisfied if we can assign to the left node x a right neighbor of length no larger than k plus a small overhead. For any $x \in L$ it is possible to have several requests $(x, k_1), (x, k_2), \dots$ and, thus, a left node x may receive as matches several right nodes. On the other hand, a right node cannot match more than one left node. For every k , there are at most 2^k requests of the form (x, k) for various $x \in L$. Assignments cannot be revoked. We will sometimes call right nodes *hash-values*.

Definition II.1. Let $c(n)$ be a function of n with natural values. A bipartite graph $G = (L, R, E \subseteq L \times R)$, whose left and right nodes are binary strings *has matching with overhead* $c(n)$ if the following holds. For every set $S \subseteq L \times \mathbb{N}$

of pairs (x, k) having at most 2^k pairs with the second component k (for all k) one can choose for every pair (x, k) in S a neighbor $p(x, k)$ of x so that $|p(x, k)| \leq k + c(|x|)$ and $p(x_1, k_1) \neq p(x_2, k_2)$ whenever $x_1 \neq x_2$. If this is done, we will say that $p(x, k)$ *matches* x .

A bipartite graph *has on-line matching with overhead* $c(n)$ if this can be done in the on-line fashion: requests for matching (x, k) appear one by one and we have to find $p(x, k)$ before the next request appears. All the made assignments cannot be changed.

Definition II.2. A bipartite graph is *computable* (*polynomial-time computable*) if given a left node x we can compute (respectively, compute in time polynomial in $|x|$) the list of all its neighbors. A polynomial-time computable graph is also said to be *explicit*.

The next two theorems show that the existence of a computable (polynomial-time computable) list approximator f for c -short programs is equivalent to the existence of a computable (polynomial-time computable) graph G that has on-line matching with overhead $c + O(1)$ and $|f(x)|$ is equal to the degree of x in G .

Theorem II.3 (Graph G with on-line matching \Rightarrow list-approximator f). *Assume there is a computable graph with $L = \{0, 1\}^*$ where the left node x has degree $D(x)$ and which has on-line matching with overhead $c(n)$. Assume further that the matching strategy is computable. Then for every standard machine U there exists a computable function f that for any x produces a list with $D(x)$ many elements containing a program p for x of length $|p| = C_U(x) + c(|x|) + O(1)$. If the graph is polynomial time computable then the function f is polynomial time computable, too.*

Proof: Run the optimal machine $U(q)$ in parallel for all strings q . Once $U(q)$ halts with the result x we pass the request $(x, |q|)$ to the matching algorithm in the graph and find a hash value p of length at most $|q| + c(|x|)$ for x .

By construction, every string x is matched to a string p of length at most $C_U(x) + c(|x|)$. Each right node is matched to at most one node in the graph. Hence there is a machine V such that $V(p) = x$ whenever p is matched to x . Thus for every string x there is a neighbor p of x with $|p| \leq C_U(x) + c(|x|)$ and $V(p) = x$. As U is a standard machine, there is a (polynomial time) computable function t with $U(t(p)) = V(p)$ and $|t(p)| \leq |p| + O(1)$. Let $f(x)$ be the list consisting of $t(p)$ for all the neighbors p of x in the graph. By construction $|f(x)| = D(x)$ and we are done. ■

Remark 1. If $c(n) = c$ is a constant function, then we can construct a standard machine U_1 which has a computable approximator for 0-short programs with lists of size $D(x)$. To this end let $U_1(0p) = V(p)$ and $U_1(1^{c+2}p) = U(p)$ and let $t(p) = 0p$ in the construction of f in the proof of the above theorem. By Theorem II.5 below there is a graph having on-line matching with constant overhead and degree $O(|x|^2)$.

Thus there is a standard machine U_1 which has a computable approximator for 0-short programs with lists of size $O(|x|^2)$.

Theorem II.4 (List-approximator $f \Rightarrow$ graph G with on-line matching). *Assume that $c(n)$ is (polynomial-time) computable function and there are an optimal machine U and a (polynomial-time) computable function f that for any x produces a finite list containing a program p for x of length $|p| \leq C_U(x) + c(|x|)$. Consider the bipartite graph G with $L = \{0, 1\}^*$ where the neighbors of node x are all strings from $f(x)$. Then G has on-line matching with overhead $c(|x|) + O(1)$.*

Proof: For each n let G_n be the subgraph of G with $L = \{0, 1\}^{\leq n}$. W.l.o.g. we assume that all strings in $f(x)$ have length at most $|x| + O(1)$ and hence the graph G_n is finite. We claim that G_n has on-line matching with overhead $c(|x|) + O(1)$ for all n , (where the $O(1)$ constant does not depend on n).

We first show that this implies the theorem. Suppose that M_1, M_2, \dots are on-line matching strategies for graphs G_1, G_2, \dots . It suffices to convert them to strategies M'_1, M'_2, \dots for G_1, G_2, \dots such that for all $i, j > i$ strategy M'_j is an extension of M'_i , i.e. on a series of requests only containing nodes from G_i , strategy M'_j behaves exactly as M'_i . Because each G_n finite, there are only finitely many different matching strategies for G_n . Hence, there is a strategy M'_1 that equals the restriction of M_n to G_1 for infinitely many n . Therefore there is also a strategy M'_2 that is an extension of M'_1 and equals the restriction of M_n to G_2 infinitely often, and so on.

It remains to show the claim. For the sake of contradiction assume that for every constant i there is n such that G_n does not have on-line matching with overhead $c(|x|) + i$, and f is a list-approximator for $c(|x|)$ -short programs on U . Because G_n is finite, for all n and c one can find algorithmically (using an exhaustive search) whether G_n has on-line matching with overhead $c(|x|) + i$ or not. One can also find a winning strategy for that player who wins (“Matcher” or “Requester”). Therefore for every i we can algorithmically find the first n such that the graph G_n does not have on-line matching with overhead $c(|x|) + i$ and the corresponding winning strategy for Requester for G_n .

Let that strategy play against the following “blind” strategy of Matcher. Receiving a request (x, k) , the Matcher runs $U(p)$ for all $p \in f(x)$, $|p| \leq k + c(|x|) + i$, in parallel. If for some p , $U(p)$ halts with the result x , he matches the first such p to x and proceeds to the next request. Otherwise the request remains not fulfilled.

Consider the following machine V . On input (q, i) , with q a k -bit long string and i a natural number, it finds the first G_n such that the graph G_n does not have on-line matching with overhead $c(|x|) + i$ and a winning strategy for Requester, and runs it against the blind strategy of the Matcher. Then it returns x , where (x, k) is the q -th request with the second

component k (we interpret the string q as the ordinal of the request in some standard manner). Since the Requester wins, there is a request (x, k) that was not fulfilled. We have

$$C_U(x) \leq C_V(x) + O(1) \leq k + 2 \log i + O(1) \leq k + i \quad (1)$$

(the last inequality holds for all large enough i). As the request (x, k) was not fulfilled, there is no p in $f(x)$ with $|p| \leq k + i + c(|x|)$. Due to (1), $f(x)$ has no $c(|x|)$ -short program for x , a contradiction. ■

We have reduced the problem of list approximation for c -short programs to the construction of bipartite graphs with on-line matching that have polynomial left degree. Our main technical contributions are the following theorems.

Theorem II.5 (Combinatorial version of Theorem I.1). *There is a computable graph with $L = \{0, 1\}^*$ with left degree $D(x) = O(|x|^2)$ which has on-line matching with overhead $O(1)$.*

Theorem II.6 (Combinatorial version of Theorem I.2). *There is a polynomial time computable graph with $L = \{0, 1\}^*$ with left degree $D(x) = \text{poly}(|x|)$ which has on-line matching with overhead $O(\log |x|)$.*

Theorem II.7 (Combinatorial version of Theorem I.3). *In every graph G with $L = \{0, 1\}^n$ that has off-line matching with overhead c , the maximal degree of left nodes is $\Omega(n^2 / (c + O(1))^2)$.*

Theorems II.5, II.6 and II.7 imply Theorems I.1, I.2 and I.3, respectively. Moreover, our on-line matching strategy used in the proofs of Theorems II.5 and II.6 are very simple: receiving a new request of the form (x, k) we just find the maximal $i \leq k + c$ such that there is a free (i.e., not yet used in a matching) right neighbor of x of length i and match x with the first such neighbor of x .

III. THE UPPER BOUNDS

In this section we prove Theorems II.5 and II.6, by constructing the corresponding graphs that have on-line matching with overhead $c(n)$. These are infinite graphs which are obtained as the union of finite bipartite graphs with $L = \{0, 1\}^n$ and $R = \{0, 1\}^{k+c(n)}$. For such graphs in which the length of the right nodes is fixed we do not need to refer to the length constraint in a matching request as in Definition II.1, and we can work with the following simpler definition.

Definition III.1. A bipartite graph has *matching up to K with at most M rejections*, if for any set of left nodes of size at most K we can drop at most M its elements so that there is a matching in the graph for the set of remaining nodes. A graph has an *on-line matching up to K with at most M rejections* if we can do this in on-line fashion. For $M = 0$ we say that the graph has *matching up to K* .

The connection between graphs with matching up to K (Definition III.1) and graphs with matching with overhead

c (Definition II.1) is the following. If a graph G has (on-line) matching with overhead c then removing from G all left nodes of length different from n and all right nodes of length more than $k+c(n)$ we obtain a graph with (on-line) matching up to 2^k . On the other hand, assume that, for some n , for all $k < n$, we have a graph $G_{n,k}$ with $L = \{0,1\}^n$ and $R = \{0,1\}^{k+c(n)}$ which has (on-line) matching up to 2^k . Then the union G_n of $G_{n,k}$ over all $k < n$ has (on-line) matching with overhead c , provided all requests (x,k) satisfy $k < |x|$. At the expense of increasing the degree and c by 1, the graph G_n can be easily modified to have (on-line) matching with overhead c unconditionally: append 0 to all right nodes of G_n and for every $x \in \{0,1\}^n$ add a new right node $x1$ connected to x only.

Thus, we will need to construct finite graphs as in Definition III.1. This will be done by constructing a certain type of expander graphs.

Definition III.2. A bipartite graph is called a (K, K') -*expander*, if every set of K left nodes has at least K' distinct neighbors.

Graphs that have off-line matching up to K are closely related to (K, K) -expanders. Indeed, any graph having matching up to K is obviously a (K', K') -expander for all $K' \leq K$. Conversely, by Hall's theorem [Hal35], any graph which is a (K', K') -expander for all $K' \leq K$ has matching up to K .

In [MRS11] it was shown that a reduction from expanders to *on-line* matching is also possible. More specifically, every family of $(2^i, 2^i)$ -expanders, one for each $i < k$, sharing the same set L of left nodes can be converted into a graph with the same set L of left nodes that has on-line matching up to 2^k , at the expense of multiplying the degree by k and increasing hash-values by 1. (We will present the construction in the proof of Theorem III.4.)

In [MRS11] it is observed that every $(2^{k-1}, 2^{k-1})$ -expander has on-line matching up to 2^k with at most 2^{k-1} rejections. We need a slight generalization of this fact.

Lemma III.3. *Every $(2^{k-\ell}, 2^k - 2^{k-\ell})$ -expander has on-line matching up to 2^k with at most $2^{k-\ell}$ rejections.*

Proof: Use the following greedy strategy for on-line matching: each time a left vertex is received, check if it has a neighbor that was not used yet. If yes, any such neighbor is selected as the match for that node. Otherwise, the node is rejected.

For the sake of contradiction, assume that the number of rejected nodes is more than $2^{k-\ell}$. Choose from them exactly $2^{k-\ell}$ rejected nodes. By the expansion property, they have at least $2^k - 2^{k-\ell}$ neighbors and all those neighbors are used by the greedy strategy (otherwise the node having a non-used neighbor would not be rejected). Thus we have at least $2^k - 2^{k-\ell}$ matched left nodes and more than $2^{k-\ell}$ rejected nodes. Consequently, we have received more than 2^k requests, contradiction. ■

In particular, every $(2^k, 2^k)$ -expander has on-line matching up to 2^{k+1} with at most 2^k rejections. For Theorem II.5 we will use non-explicit such graphs, for Theorem II.6 we will need explicit such graphs, which we obtain from the disperser of [TSUZ07].

Assume that for every n and $k < n$ we are given a $(2^k, 2^k)$ -expander $G_{n,k}$ with $L = \{0,1\}^n$, $R = \{0,1\}^{k+c(n)}$ and the degree of all left nodes is at most $D(n)$. Assume further that, given n, k and a left node x in $G_{n,k}$, we can algorithmically find the list of all neighbors of x in $G_{n,k}$.

Theorem III.4. *Given a family of expanders as above we can construct a computable graph G with $L = \{0,1\}^*$ that has on-line matching with overhead $c(n) + O(\log n)$ and the degree of each left node is $O(D(n)n)$. The matching strategy for G is computable. Moreover, if given n, k and a left node x in $G_{n,k}$ we can find the list of all neighbors of x in time $\text{poly}(n)$ then G is polynomial time computable.*

Proof: The main tool is borrowed from [MRS11]: all the graphs $G_{n,k}$ share the same set of left nodes while their sets of right nodes are disjoint. Let $H_{n,k}$ denote the union of $G_{n,i}$ over all $i < k$. Then $H_{n,k}$ has on-line matching up to 2^k (without rejections). Indeed, each input left node is first given to the matching algorithm for $G_{n,k-1}$ (that has on-line matching up to 2^k with at most 2^{k-1} rejections) and, if rejected is given to the matching algorithm for $G_{n,k-2}$ and so on.

Using this construction we can prove the theorem with slightly worse parameters as claimed. To this end identify right nodes of the graph $H_{n,k}$ with strings of length $k+c(n)+1$ (the number of right nodes of $H_{n,k}$ does not exceed the sum of geometrical series $2^{k+c(n)} + 2^{k+c(n)-1} + \dots < 2^{k+c(n)+1}$). The degree of $H_{n,k}$ is $D(n)k$.

Recall the connection between matching up to 2^k and matching with overhead (the paragraph after Definition III.1). We see that the family $H_{n,k}$ can be converted into a graph H_n with $L = \{0,1\}^n$ and degree $D(n)n(n-1)/2+1$ having on-line matching with overhead $c(n)+2$. Finally, we prefix the right nodes of H_n with the $O(\log n)$ -bit code of the number n and consider the union of all H_n . The resulting graph has on-line matching with overhead $c(n) + O(\log n)$, its set of left nodes is $\{0,1\}^*$ and the degree of every left node of length n is $O(D(n)n^2)$.

Now we will explain how to reduce the degree to $O(D(n)n)$. Consider four copies of $G_{n,k-1}$ with the same set L of left nodes and disjoint sets of right nodes (say append 00 to every right node to get the first copy, 01 to get the second copy and so on). Their union is a $(2^{k-1}, 2^{k+1})$ -expander, and hence has matching up to 2^{k+1} with at most 2^{k-1} rejections.⁵ Its left degree is $4D(n)$ and the length of right nodes is $k+c(n)+2$. Replace in the above construction

⁵One can also consider the union of $G_{n,k-1}$ and $G_{n,k}$, which also has matching up to 2^{k+1} with at most 2^{k-1} rejections.

of H_n the graph $H_{n,k}$ by this graph. Thus the left degree of H_n becomes $O(D(n)n)$ in place of $O(D(n)n^2)$. It remains to show that (the union of all graphs) H_n has still on-line matching with overhead $c(n) + O(\log n)$

Again the matching strategy is greedy. Once we receive a request (x, k) with $|x| = n$, we match x to $1x$ if $k \geq n$. Otherwise, we pass x to the matching algorithm in $H_{n,k}$. If the algorithm rejects x , we pass x to the matching algorithm in $H_{n,k-1}$ and so on. We claim that we eventually find a match in one of the graphs $H_{n,i}$ for $i \leq k$. To prove the claim it suffices to show that the matching algorithm for $H_{n,k}$ receives at most 2^{k+1} input strings. This is proved by a downward induction on k (for any fixed n). For the base case, $k = n - 1$, this is obvious: we try to match in $H_{n,n-1}$ up to 2^{n-1} strings. The induction step: by induction hypothesis the matching algorithm for $H_{n,k+1}$ receives at most 2^{k+2} input strings and thus rejects at most 2^k of them. The matching algorithm for $H_{n,k}$ thus receives at most 2^k rejected strings and at most 2^k new ones, coming from requests of the form (x, k) . ■

A. Proof of Theorem II.5

A weaker form of Theorem II.5 can be derived from Theorem III.4 and the following lemma from [Muc02].

Lemma III.5. *For all n and $k < n$, there exists a $(2^k, 2^k)$ -expander with $L = \{0, 1\}^n$, $R = \{0, 1\}^{k+2}$ and all left nodes have degree at most $n + 1$.*

Proof: We use the probabilistic method, and for each left node we choose its $n + 1$ neighbors at random: all neighbors of each node are selected independently among all 2^{k+2} right nodes with uniform distribution, and the choices for different left nodes are independent too. We show that the expansion property is satisfied with positive probability. Hence there exists at least one such graph. To estimate the probability that the property is not satisfied, consider a pair of sets L' and R' of left and right nodes, respectively, of sizes $2^k, 2^k - 1$. The probability that the neighbors of all nodes in L' belong to R' is upper-bounded by $(1/4)^{(n+1)2^k}$. The total probability that the expansion condition is not satisfied, is obtained by summing over all such L', R' , i.e.

$$\begin{aligned} \left(\frac{1}{4}\right)^{(n+1)2^k} (2^n)^{2^k} (2^{k+2})^{2^k-1} &\leq \left(\frac{2^n 2^{k+2}}{4^{n+1}}\right)^{2^k} \\ &\leq \left(\frac{2^n 2^{n+1}}{4^{n+1}}\right)^{2^k} \\ &= \left(\frac{1}{2}\right)^{2^k} \\ &< 1. \end{aligned}$$

Remark 2. By the very same construction we can obtain a graph with $L = \{0, 1\}^n$, $R = \{0, 1\}^{k+2}$, $D = n + 1$ that is a (t, t) -expander for all $t \leq 2^k$. Indeed, the probability that a random graph is not a (t, t) -expander is at most $(\frac{1}{2})^t$ (we may replace 2^k by t in the above formulas). By the union

bound, the probability that this happens for some $t \leq 2^k$ is at most the sum of the geometric series $\sum_{t=1}^{2^k} (\frac{1}{2})^t < 1$. By Hall's theorem, this graph has off-line matching up to 2^k . An interesting open question is whether there is a graph with the same parameters, i.e., $L = \{0, 1\}^n$, $R = \{0, 1\}^{k+O(1)}$, $D = O(n)$, that has on-line matching up to 2^k .

From this lemma and Theorem III.4 we obtain a computable graph with on-line matching with overhead $O(\log n)$, degree $O(|x|^2)$ and $L = \{0, 1\}^*$. We now need to replace the $O(\log n)$ overhead by $O(1)$. Recall that the $O(\log n)$ appeared from the prefix code of n added to the hash values. To get rid of it we need a computable graph F_k in place of the previously used $G_{n,k}$ with the same parameters but with $L = \{0, 1\}^{\geq k}$, and not $L = \{0, 1\}^n$. Such a graph is constructed in the following lemma.

Lemma III.6. *For every k there is a computable bipartite graph F_k with $L = \{0, 1\}^{\geq k}$, $R = \{0, 1\}^{k+3}$ that is a $(2^k, 2^k)$ -expander and the degree of every left node x is $O(|x|)$.*

Proof: We first build such a graph with left nodes being all strings of length between k and $K = 2^{k+3}$. This is again done by the probabilistic method: we choose $|x| + 3$ neighbors of every node x independently. Let L_i stand for all left nodes of length i . For any $i \in k, \dots, K$, the probability that all elements of a fixed $L' \subset L_i$ are mapped to a fixed set of size at most $2^k - 1$ at the right is at most $(\frac{1}{2^3})^{(i+3)|L'|}$. The probability that some t_i elements in L_i are mapped into a fixed set of $2^k - 1$ elements at the right is bounded by

$$\begin{aligned} 2^{t_i} \left(\frac{1}{2^3}\right)^{(i+3)t_i} &= \left(\frac{1}{2^2}\right)^{t_i} \left(\frac{1}{2^3}\right)^{3t_i} \\ &\leq \left(\frac{1}{2^2}\right)^{(i+3)t_i+t_i} \\ &\leq \left(\frac{1}{2^2}\right)^{(k+3)t_i+t_i} \\ &= \left(\frac{1}{K}\right)^{2t_i} \left(\frac{1}{2}\right)^{2t_i}. \end{aligned}$$

If $\sum_{i=k}^K t_i = t$ with $t = 2^k$, the probability that the union of neighbors of t_k elements in L_k , t_{k+1} elements in L_{k+1} , ..., and t_K elements in L_K are mapped to a fixed set of size at most $2^k - 1$ is bounded by $\prod_i (\frac{1}{2K})^{2t_i} = (\frac{1}{2K})^{2t}$. Multiplying by the number $K^{2^k-1} \leq K^t$ of different right sets of size $K - 1$, and multiplying by the upper bound K^t for the number of solutions to the equation $\sum_{i=k}^K t_i = K$, we find

$$\left(\frac{1}{2K}\right)^{2t} K^t K^t \leq \left(\frac{1}{4}\right)^t < 1.$$

Hence, the total probability to randomly generate a graph that is not an expander is strictly less than 1. Therefore, a graph satisfying the conditions must exist, and can be found by exhaustive search.

On the left side, we now need to add the strings of length larger than $K = 2^{k+3}$. These nodes are connected to all the nodes on the right side. Thus the degree of every such node x is $2^k \leq |x|/2^3 \leq O(|x|)$ and we are done. ■

Remark 3. By the very same construction we can obtain a graph with $L = \{0, 1\}^{\geq k}$, $R = \{0, 1\}^{k+3}$, $D = O(n)$ that is a (t, t) -expander for all $t \leq 2^k$ and thus has off-line matching up to 2^k (use the union bound over all $t \in \{k, \dots, K\}$). An interesting open question is whether there is a graph with the same parameters that has *on-line* matching up to 2^k .

Now we can finish the proof of Theorem II.5. Appending all 2-bit strings to all the right nodes of the graph F_{k-1} (and thus increasing the degree 4 times) we obtain a $(2^{k-1}, 2^{k+1})$ -expander H_k . The union of H_k over all k is a computable graph, whose left degree is $O(|x|^2)$, and the set of left nodes is $\{0, 1\}^*$. It has on-line matching with constant overhead. This is proved by downward induction, as in Theorem III.4. Indeed, consider the step when the s -th request for matching arrives. By a downward induction on k we can again prove that the number of matching requests in H_k is at most 2^{k+1} . Now the base of induction is the maximal k for which there has been at least one request for matching in H_k among the s requests so far. We conclude that the s -th request is satisfied and, since this holds for every s , we are done.

B. Proof of Theorem II.6

By Theorem III.4, we have to construct for every $k \leq n$ an explicit $(2^k, 2^k)$ -expander of left degree $\text{poly}(n)$, with 2^n left nodes and $\text{poly}(n)2^k$ right nodes. (Recall that a graph is *explicit* if there is an algorithm that on input $x \in \{0, 1\}^n = L$ lists in $\text{poly}(n)$ time all the neighbors of x .)

The proof relies on the explicit disperser graphs of Ta-Shma, Umans, and Zuckerman from Theorem III.8 below.

Definition III.7. A bipartite graph $G = (L, R, E)$ is a (K, δ) -disperser, if every subset $B \subseteq L$ with $|B| \geq K$ has at least $(1 - \delta)|R|$ distinct neighbors.

Theorem III.8. [Ta-Shma, Umans, Zuckerman [TSUZ07]] For every K, n and constant δ , there exists an explicit (K, δ) -disperser $G = (L = \{0, 1\}^n, R = \{0, 1\}^m, E \subseteq L \times R)$ in which every node in L has degree $D = \text{poly}(n)$ and $|R| = \frac{\alpha KD}{n^3}$, for some constant α .

Given n and k , we apply this theorem to $K = 2^k$ and $\delta = 1/2$. We obtain a $(2^k, \frac{\alpha 2^k D}{2n^3})$ -expander with degree $D = \text{poly}(n)$, $L = \{0, 1\}^n$ and $|R| = \frac{\alpha KD}{n^3}$. Consider $t = \max\{1, \lceil \frac{2n^3}{\alpha D} \rceil\}$ disjoint copies of this graph and identify left nodes of the resulting graphs (keeping their sets of right nodes disjoint). We get an explicit $(2^k, 2^k)$ -expander with 2^n left and $2^k \text{poly}(n)$ right nodes and degree $\text{poly}(n)t = \text{poly}(n)$.

IV. THE LOWER BOUNDS

We first note that a weaker linear lower bound can be easily derived from the result from [BBF⁺06] cited in the Introduction. Indeed, if there were a computable list-approximator for c -short programs of x with list of size s , then we could algorithmically produce a list of size

sc containing $C(x)$. A linear lower bound also follows from a result of Bauwens [Bau12], improving a theorem of Gács [Gac74]. The result states that $C_U(C_U(x) | x)$ is greater than $\log|x| - O(1)$ for infinitely many x . Thus, for infinitely many x ,

$$\log|x| - O(1) \leq C_U(C_U(x) | x) \leq \log|f(x)| + 2\log c + O(1),$$

and therefore $|f(x)| \geq \Omega(|x|/c^2)$.

A. Proof of Theorem II.7

Assume that G has off-line matching with overhead c . Let $G[\ell, k]$ denote the induced graph that is obtained from G by removing all right nodes of length more than k or less than ℓ . The graph $G[0, k+c]$ is obviously a $(2^k, 2^k)$ -expander for every k . As there are less than 2^{k-1} strings of length less than $k-1$, it follows that the graph $G[k-1, k+c]$ is a $(2^k, 2^{k-1} + 1)$ -expander.

The next lemma inspired by [KST54] (see [RTS00, Theorem 1.5]) shows that any such expander must have large degree.

Lemma IV.1. Assume that a bipartite graph with 2^ℓ left nodes and 2^{k+c} right nodes is a $(2^k, 2^{k-1} + 1)$ -expander. Then there is a left node in the graph with degree more than $D = \min\{2^{k-2}, (\ell - k)/(c + 2)\}$.

Proof: For the sake of contradiction assume that all left nodes have degree at most D (and w.l.o.g. we may assume that all degrees are exactly D). We need to find a set of right nodes B of size 2^{k-1} and 2^k left nodes all of whose neighbors lie in B . The set B is constructed via a probabilistic construction. Namely, choose B at random (all $\binom{2^{k+c}}{2^{k-1}}$ sets have equal probabilities). The probability that all neighbors of a fixed left node are in B is equal to

$$\frac{\binom{2^{k+c}-D}{2^{k-1}-D}}{\binom{2^{k+c}}{2^{k-1}}} = \frac{2^{k-1}(2^{k-1}-1)\dots(2^{k-1}-D+1)}{2^{k+c}(2^{k+c}-1)\dots(2^{k+c}-D+1)}.$$

Both products in the numerator and denominator have D factors and the ratio of corresponding factors is at least

$$\frac{2^{k-1}-D+1}{2^{k+c}-D+1} \geq 2^{-c-2}$$

(the last inequality is due to the assumption $D \leq 2^{k-2}$). Thus the probability that all neighbors of a fixed left node are in B is at least $2^{-D(c+2)}$. Hence the average number of left nodes having this property is at least $2^{\ell-D(c+2)}$, which is greater than or equal to 2^k by the choice of D . Hence there is B that includes neighborhoods of at least 2^k left nodes, a contradiction. ■

This lemma states that at least one left node has large degree. However, it implies more: if the number of left nodes is much larger than 2^ℓ , then almost all left nodes must have large degree. Indeed, assume that a bipartite graph with 2^{k+c} right nodes is a $(2^k, 2^{k-1} + 1)$ -expander. Choose 2^ℓ left nodes

with smallest degree and apply the lemma to the resulting induced graph (which is also a $(2^k, 2^{k-1} + 1)$ -expander). By the lemma, in the original graph all except for less than 2^ℓ nodes have degree more than $D = \min\{2^{k-2}, (\ell - k)/(c + 2)\}$.

Choose $n/4 < k \leq n/2$. As noticed, the graph $G[k-1, k+c]$ is a $(2^k, 2^{k-1} + 1)$ -expander and has less than 2^{k+c+1} right nodes. By Lemma IV.1 and the above observation (applied to $\ell = 3n/4$), all except for at most $2^{3n/4}$ left nodes of $G[k-1, k+c]$ have degree at least $n/(4(c+3))$.

Pick now ℓ different integers k_i with $n/4 < k_1 < k_2 < \dots < k_\ell < n/2$ that are $c+2$ apart of each other, where ℓ is about $n/(4(c+2))$. In each graph $G[k_i-1, k_i+c]$, all the left nodes, except at most $2^{3n/4}$, have degree $\geq n/(4(c+3))$. Since the k_i are $c+2$ apart, and thus the graphs $G[k_i-1, k_i+c]$ have pairwise disjoint right sets, it follows that G has left nodes with degree $\Omega(n^2/(c+3)^2)$.

B. Proof of Theorem 1.4.

The size of list-approximators is closely related to total conditional Kolmogorov complexity, which was first introduced by A. Muchnik and used in [Ver09], [Bau10]. *Total conditional Kolmogorov complexity* with respect to U is defined as:

$$CT_U(u|v) = \min \{ |q| : U(q, v) = u \wedge \forall z [U(q, z) \downarrow] \},$$

where $U(q, z) \downarrow$ means that $U(q, z)$ halts. If U is a standard machine then $CT_U(u|v) \leq CT_V(u|v) + c_V$ for every machine V . The connection to list-approximators is the following:

Lemma IV.2. *If f is a computable function that maps every string to a finite list of strings then $CT_U(p|x) \leq \log |f(x)| + O(1)$ for any standard machine U and every p in $f(x)$. The constant in O -notation depends on f and U .*

Proof: Let $V(j, x)$ stand for the j th entry of the list $f(x)$, if $j \leq |f(x)|$, and for the (say) empty string otherwise. Obviously $CT_V(p|x) \leq \log |f(x)|$ for all p in $f(x)$. Hence $CT_U(p|x) \leq \log |f(x)| + O(1)$. ■

Thus to prove the theorem it suffices to construct a standard machine U_0 such that, for infinitely many x , every 0-short p for x with respect to U_0 satisfies $CT_{U_0}(p|x) \geq |x| - O(1)$. To this end we fix any standard machine U and construct another machine V such that for some constant d and for every integer k there are strings p, x such that:

- (a) p is the unique 0-short program for x with respect to V ,
- (b) $C_U(x) \geq k$,
- (c) $|x| = |p| = k + d$,
- (d) $CT_U(0p|x) \geq k$.

Once such V has been constructed, we let $U_0(0q) = V(q)$ and $U_0(1^{d+2}q) = U(q)$. The latter equality guarantees that U_0 is a standard machine. And both equalities together with items (a), (b) and (c) imply that $0p$ is the unique 0-short program for x with respect to U_0 . Finally, item (d) guarantees that its total complexity conditional to x is at least $|x| - d$.

The construction of V can be described in game terms.

Description of the game. The game has integer parameters k, d and is played on a rectangular grid with 2^{k+d} rows and 2^{k+d} columns. The rows and columns are identified with strings of length $k+d$. Two players, Black and White, play in turn. In her turn White can either pass or put a pawn on the board. White can place at most one pawn in each row and at most one pawn in each column. Once a pawn is placed, it can not be moved nor removed. In his turn Black can either pass, or choose a column and *disable* all its cells, or choose at most one cell in every column and *disable* all of them. If a player does not pass, we say that she/he *makes a move*. Black is allowed to make less than 2^{k+1} moves. The game is played for an infinite time and White loses if at some point after her turn, all her pawns are in disabled cells.

We will show that, for $d = 3$, for every k , White wins this game. More specifically, there is a winning strategy for White that is uniformly computable given k . Assume that this is done. Then consider the following “blind” strategy for Black: start enumeration of all strings x with $C_U(x) < k$ and all strings q of length less than k such that $U(q, x) \downarrow$ for all x of length $k+d$. That enumeration can be done uniformly in k . In his t th turn Black: disables all cells in the x th column, if on step t in this enumeration a new x of length $k+d$ with $C_U(x) < k$ appears; disables all cells (p, x) with $|x| = |p| = k+d$, $U(q, x) = 0p$, if on step t a new string q of length less than k appears such that $U(q, x) \downarrow$ for all x of length $k+d$; and passes if none of these events occurs. Note that the total number of Black’s moves is less than $2^k + 2^k = 2^{k+1}$, as required.

Now consider the following machine $V(p)$: let $k = |p| - d$ and let the White’s computable winning strategy play against Black’s blind strategy. Watch the play waiting until White places a pawn on a cell (p, x) in the p -th row. Then output x and halt. Note that such x is unique (if exists), as White places at most one pawn in each row. Because White’s strategy is winning, at least one of the finitely many pawns in a cell (p, x) will never be disabled. This pair (p, x) satisfies all the requirements (a)–(d). Thus it suffices to design a computable winning strategy for White.

A winning White’s strategy. The strategy is a greedy one. In the first round White places a pawn in any cell. Then she waits until that cell becomes disabled. Then she places the second pawn in any enabled cell that lies in another row and another column and again waits until that cell becomes disabled. At any time she chooses any enabled cell that lies in a row and a column that both are free of pawns. In order to show that White wins, we just need to prove that there is such a cell. Indeed, Black makes less than 2^{k+1} moves, thus White makes at most 2^{k+1} moves. On each of Black’s moves at most 2^{k+d} cells become disabled. On each of White’s moves at most 2^{k+d+1} cells becomes non-free because either their column or row already has a pawn. Thus if the total

number of cells is more than

$$2^{k+1}2^{k+d} + 2^{k+1}2^{k+d+1} = 6 \cdot 2^{2k}2^d,$$

we are done. The total number of cells is $2^{k+d}2^{k+d} = 2^{2k}2^{2d}$. As 2^{2d} grows faster than $6 \cdot 2^d$, for large enough d (actually for $d = 3$) the total number of cells is larger than the number of disabled or non-free cells. The theorem is proved.

Note that changing the construction a little bit we can prove the same statement for c -short programs for every c . To this end we just need to let $U_0(1^{c+d+2}q) = U(q)$ instead of $U_0(1^{d+2}q) = U(q)$. The optimal machine U_0 constructed in this way depends on the choice of c , which is inevitable by Theorem I.1.

V. OTHER APPLICATIONS OF EXPLICIT GRAPHS WITH ON-LINE MATCHING

Our applications are related to the resource bounded Kolmogorov complexity. Recall that a machine U is called standard if for any machine V there is a total computable function f such that $U(t(p), z) = V(p, z)$ and $|t(p)| \leq |p| + O(1)$ for all p, z . In this section we assume that t is polynomial-time computable and that running time of $U(t(p), z)$ is bounded by a polynomial of the computation time of $V(p, z)$. By $CD_U^f(x|z)$ we denote the minimal length of p such that $U(p, z) = x$ in at most T steps.

The applications consists in improving Muchnik's Theorem, and the error term in the estimation of the distinguishing complexity of strings in a given set.

A. Muchnik's theorem

Theorem V.1 (Muchnik's Theorem [Muc02], [MRS11], [Mus11]). *Let a and b be strings such that $|a| = n$ and $C(a | b) = k$. Then there exists a string p such that (1) $|p| = k + O(\log n)$, (2) $C(p | a) = O(\log n)$, (3) $C(a | p, b) = O(\log n)$.*

See the cited works for a discussion of Muchnik's theorem.

Theorem V.2 (Improved version of Theorem V.1). *Same statement as above except that we replace (2) by (2') $C^{q(n)}(p | a) = O(\log n)$, where q is a polynomial.*

Proof: Fix an explicit graph $G = (L, R, E)$, with $L = \{0, 1\}^*$, polynomial left degree, and that has computable on-line matching with logarithmic overhead (such a graph is obtained in the proof of Theorem II.6). Given a string b , run the optimal machine $U(q, b)$ in parallel for all q . Once, for some q , $U(q, b)$ halts with the result x , pass the request $(x, |q|)$ to the matching algorithm in the graph. It will return a neighbor p of length at most $|q| + O(\log n)$ of x . At some moment a shortest program q for a conditional to b will halt and we get the sought p .

As the graph is explicit and has polynomial degree, we have $C^{\text{poly}(n)}(p | a) = O(\log n)$ (requirement (2')). Requirement (1) holds by construction. Finally, $C(a | p, b) =$

$O(\log n)$ as given p and b we may identify a by running the above algorithmic process (it is important that a is the unique string that was matched to p). ■

B. Distinguishing complexity.

The T -bounded distinguishing complexity of a string x , introduced by Sipser [Sip83], is the length of a minimal program that in T steps accepts x and only x . Formally, let U be a machine, x a string and T a natural number. The *distinguishing complexity* $CD_U^T(x)$ with respect to U is defined as the minimal length of p such that $U(p, x) = 1$ (p "accepts" x) in at most T steps, and $U(p, x') = 0$ for all $x' \neq x$ (p "rejects" all other strings). From our assumption for the standard machine U , it follows that for every machine V there is a polynomial f and a constant c such that $CD_U^{f(T)}(x) \leq CD_V^T(x) + c$. Indeed, let p be a shortest distinguishing program for x working in T steps with respect to V . Then $t(p)$ is a program for U that accepts x in $\text{poly}(T)$ steps and rejects all other strings.

For a set A of binary strings, let $A^{=n}$ stand for the set of all strings of length n in A . Buhrman, Fortnow and Laplante [BFL01] have shown that most strings have polynomial-time distinguishing complexity close to the information-theoretic minimum value.

Theorem V.3 ([BFL01]). *For every function $\varepsilon(n)$ (mapping natural numbers to numbers of the form $1/\text{natural}$) computable in time $\text{poly}(n)$ there is a polynomial f such that for every set A , for all $x \in A^{=n}$ except for a fraction $\varepsilon(n)$, $CD_U^{f(n), A}(x) \leq \log |A^{=n}| + \text{polylog}(n/\varepsilon(n))$.*

We mean here that the set A is given to the standard machine U as an oracle (so we assume that the standard machine is an oracle machine and all the requirements hold for every oracle.)

Our improvement of the above theorem does not use explicitly graphs with on-line matching. However, it uses an argument similar to the one in Theorem II.6 to obtain a graph with a certain "low-congestion" property, which would allow most nodes making matching requests to have their "reserved" matching node.

Theorem V.4 (Improved version of Theorem V.3). *Same statement as above, except that we obtain $CD^{f(n), A}(x) \leq \log |A^{=n}| + O(\log(n/\varepsilon(n)))$, i.e., we reduce the error term from $\text{polylog}(n/\varepsilon(n))$ to $O(\log(n/\varepsilon(n)))$.*

Proof: For our improvement we need for every n , $k \leq n$ and ε a bipartite graph $G_{n, k, \varepsilon}$ with $L = \{0, 1\}^n$, $R = \{0, 1\}^{k+O(\log n/\varepsilon)}$ and degree $\text{poly}(n/\varepsilon)$ that has the following "low-congestion" property:

for every subset S of at most 2^k left nodes for every node x in S except for a fraction ε there is a right neighbor p of x such that p has no other neighbors in S .

Assume that we have such an explicit family of graphs $G_{n,k,\varepsilon}$. Here, explicit means that given n, k, ε , a left node x and i , we can in polynomial time find the i -th neighbor of x in $G_{n,k,\varepsilon}$. Then we can construct a machine V that, given a tuple (p, i, n, k) , a string x and A as oracle, verifies that x is in $A^{\neg n}$ and that x is the i -th neighbor of p in $G_{[x],k,\varepsilon(n)}$. If this is the case it accepts and rejects otherwise. By the property of the graph, applied to $S = A^{\neg n}$ and $k = \lceil \log |S| \rceil$ we see that

$$CD_V^{f(n),A}(x) \leq |(p, i, n, k)| \leq \log |A^{\neg n}| + O(\log n / \varepsilon(n))$$

for some polynomial $f(n)$ for all but a fraction $\varepsilon(n)$ for $x \in A^{\neg n}$. By the assumptions on U , the same inequality holds for U .

The graph $G_{n,k,\varepsilon}$ is again obtained from the disperser of [TSUZ07]. Given n, k and ε , we apply Theorem III.8 to $K = \varepsilon 2^k$ and $\delta = 1/2$. We obtain a $(\varepsilon K, \frac{\alpha \varepsilon K D}{2n^3})$ -expander with degree $D = \text{poly}(n)$, $L = \{0, 1\}^n$ and $|R| = \frac{\alpha K D}{n^3}$. Consider $t = \max\{1, \lceil \frac{2n^3}{\alpha D} \rceil\}$ disjoint copies of this graph and identify left nodes of the resulting graphs (keeping their sets of right nodes disjoint). We get an explicit $(2^k \varepsilon, 2^k \varepsilon)$ -expander with 2^n left and $2^k \text{poly}(n) / \varepsilon$ right nodes and degree $D = \text{poly}(n)t / \varepsilon = \text{poly}(n) / \varepsilon$.

This graph, called $H_{n,k,\varepsilon}$, has the following weaker “low-congestion” property: *for every set of 2^k left nodes S for every node x in S except for a fraction ε there is a right neighbor p of x such that p has at most D/ε neighbors in S .*

Indeed, the total number of edges in the graph originating in S is at most $|S|D$. Thus less than $|S|D/(D/\varepsilon) = |S|\varepsilon$ right nodes are “fat” in the sense that they have more than D/ε neighbors landing in S . By the expander property of $H_{n,k,\varepsilon}$ there are less than $\varepsilon|S|$ left nodes in S that have only fat neighbors.

It remains to “split” right nodes of $H_{n,k,\varepsilon}$ so that D/ε becomes 1. This is done exactly as in [BFL01]. Using the Prime Number Theorem, it is not hard to show (Lemma 3 in [BFL01]) that for every set W of d strings of length n the following holds: *for every $x \in W$ there is a prime number $q \leq 4dn^2$ such that $x \not\equiv x' \pmod{q}$ for all $x' \in W$ different from x* (we identify here natural numbers and their binary expansions).

We apply this lemma to $d = D/\varepsilon$. To every right node p in $H_{n,k,\varepsilon}$ we add a prefix code of two natural numbers a, q , both at most $4dn^2$, and connect a left node x to (p, a, q) if x is connected to p in $H_{n,k,\varepsilon}$ and $x \equiv a \pmod{q}$. We obtain the graph $G_{n,k,\varepsilon}$ we were looking for. Indeed, for every S of 2^k left nodes for all $x \in S$ but a fraction of ε there is a neighbor p of x in $H_{n,k,\varepsilon}$ that has at most $d = D/\varepsilon = \text{poly}(n) / \varepsilon$ neighbors in S . Besides there is a prime $q \leq 4n^2 d = \text{poly}(n) / \varepsilon$ such that $x \not\equiv x' \pmod{q}$ for all neighbors x' of p different from x . Thus the neighbor $(p, q, x \bmod q)$ of x in $G_{n,k,\varepsilon}$ has no other neighbors in S .

The degree of $G_{n,k,\varepsilon}$ is $D \times (4n^2 D / \varepsilon)^2 = \text{poly}(n) / \varepsilon^2$. The number of right nodes is

$$(\text{poly}(n) 2^k / \varepsilon) (4n^2 D / \varepsilon)^2 = 2^k \text{poly}(n) / \varepsilon^3.$$

Thus right nodes can be identified with strings of length $k + O(\log n / \varepsilon)$ and we are done. ■

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