Duality in Abstract Definability Theory

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Abstract definability theory

Consider:

- $\mathcal{L}$ — a language (its elements are called formulas)
- $\mathcal{M}$ — a class of models (or structures)
- $\models$ — a truth relation: $M \models A$ between $M \in \mathcal{M}$ and $A \in \mathcal{L}$

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- classes of models from $\mathcal{M}$ definable by a single formula from $\mathcal{L}$?
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For a set of formulas $\Gamma \subseteq \mathcal{L}$ and a class of models $\mathbb{K} \subseteq \mathcal{M}$, we denote:

$$\begin{array}{lcl} \mathrm{Models}(\Gamma) & := & \{ M \in \mathcal{M} \mid M \models \Gamma \} \\ \mathrm{Theory}(\mathbb{K}) & := & \{ A \in \mathcal{L} \mid \mathbb{K} \models A \} \end{array}$$

The 4 “species” of classes

Definition. For a class of models $\mathbb{K} \subseteq \mathcal{M}$, we write:

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Equivalently: if $\mathbb{K} = \bigcap\limits_{i \in I} \mathbb{K}_i$ for some classes $\mathbb{K}_i \in \mathbb{L}$.

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For the "elementary" (i.e. first-order) language $\mathcal{L}$, the terminology is:

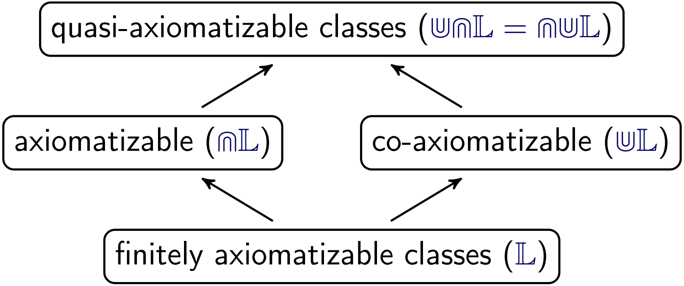
$\mathbb{K} \in \mathbb{L}$ — an elementary class of models (or *finitely axiomatizable*)

$\mathbb{K} \in \Cap\mathbb{L}$ — a Δ-elementary class of models (or *axiomatizable*)

$\mathbb{K} \in \Cup\mathbb{L}$ — a Σ-elementary class of models (or *co-axiomatizable?*)

$\mathbb{K} \in \Cup\Cap\mathbb{L}$ — a $\Sigma\Delta$-elementary class of models (or *quasi-axiomatizable*?)

The hierarchy of the 4 species of classes of models



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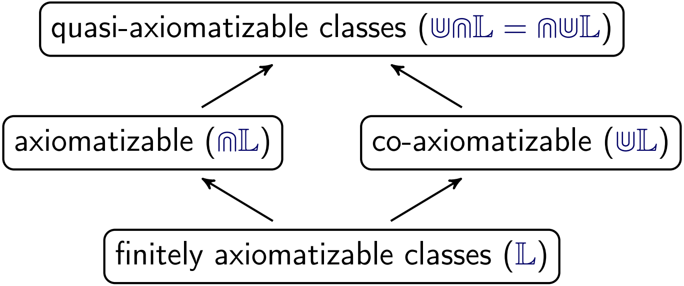

- Classes in $\mathbb{L}$: the classes of all groups, all rings, all fields, all orders
- Classes in $\Cap\mathbb{L}$: infinite groups, infinite rings, infinite fields, inf. orders
- Classes in $\Cup\mathbb{L}$: finite groups, finite rings, finite fields, finite orders
- Classes in $\Cup\Cap\mathbb{L}$: infinite fields of characteristic $p > 0$;
  infinite finitely dimensional vector spaces
- Not even in $\Cup\Cap\mathbb{L}$: well-ordered sets, periodic groups, simple groups

Characterization of quasi-axiomatizable classes of models

Definition

For models $M, N \in \mathcal{M}$, we write $M \sqsubseteq N$ if,

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Theorem (E.Z., 2017)

For any class of models $\mathbb{K} \subseteq \mathcal{M}$, the following conditions are equivalent:

(A1) $\mathbb{K}$ *is the union of intersections of finitely ax. classes:* $\mathbb{K} \in \Cup\Cap\mathbb{L}$

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(C1) $\mathbb{K} = \bigcup\limits_{M \in \mathbb{K}} \bigcap\limits_{M \models A} \mathrm{Models}(A)$

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Characterization of quasi-axiomatizable classes (improved)

Notation: all theories of models M from the class $\mathbb{K}$:

$$\mathrm{TH}(\mathbb{K}) \quad := \quad \{\, \mathrm{Theory}(M) \mid M \in \mathbb{K} \,\} \qquad \text{— this is a set!}$$

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Duality of notions in model theory

models of a formula $A \in \mathcal{L}$:	the theory of a model $M \in \mathcal{M}$:
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Theorem. Conditions (1) and (2) are equivalent!

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Examples:

First-order logic, group theory, modal logic **K** — are theories

True arithmetic — $\text{Theory}(\mathbb{N}, +, \times, =)$ — is a basic theory

Characterization of quasi-theories

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As before, we can avoid unions and intersection over *classes*.

Compactness

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Examples of compact classes:
– all groups, linear orders, Boolean algebras, etc.
– all infinite groups, linear orders, Boolean algebras, etc.
– all reflexive Kripke frames or models, etc.

Examples of non-compact class: all finite groups; all well-orders, etc.

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Definition (Truth and Boolean connectives)

- The truth relation $\models$ respects negation if

$$M \not\models A \quad \Longleftrightarrow \quad M \models \neg A$$

- The truth relation $\models$ respects conjunction if

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Remark. If $\models$ respects negation, then $M \sqsubseteq N$ is equivalent to $M \equiv N$.

Characterization of axiomatizable classes

Theorem (E.Z., 2017)

Suppose that the system $(\mathcal{M}, \models, \mathcal{L})$ *is such that*

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Then, for any class of models $\mathbb{K} \subseteq \mathcal{M}$*, the following holds:*

- $\mathbb{K}$ *is axiomatizable* $\iff$ $\mathbb{K}$ *is closed under* $\equiv$ *and compact*
- $\mathbb{K}$ *is co-axiomatizable* $\iff$ $\overline{\mathbb{K}}$ *is closed under* $\equiv$ *and compact*
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	Both	$\mathbb{K}$ is ...	$\overline{\mathbb{K}}$ is ...
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$\mathbb{K} \in \Cap\mathbb{L}$ ($\mathbb{K}$ is axiomatizable)	$\equiv$	compact	
$\mathbb{K} \in \mathbb{L}$ ($\mathbb{K}$ is finitely axiomatizable)	$\equiv$	compact	compact

Conclusion and further directions

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- In the relation $M \models A$ there is a natural "symmetry" between the objects on the left (models) and the objects on the right (formulas).
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- One can introduce notions and obtain results that are dual to the known ones: exchange the role of models and formulas (if possible).
- But there are subtleties:
  – usually models form a class,
  – but formulas form a set.
- Therefore, the main (philosophical) question is: *For which notions and results, the dual notions and results are meaningful and interesting?*

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- In the relation $M \models A$ there is a natural "symmetry" between the objects on the left (models) and the objects on the right (formulas).
- One can introduce notions and obtain results that are dual to the known ones: exchange the role of models and formulas (if possible).
- But there are subtleties:
  – usually models form a class,
  – but formulas form a set.
- Therefore, the main (philosophical) question is: *For which notions and results, the dual notions and results are meaningful and interesting?*

More technically:

- What is the notion dual to that of a compact class of models?
- What are criteria for $T \subseteq \mathcal{L}$ to be a theory, co-theory, basic theory?

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Thank you!